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Procedia Engineering 00 (2017) 000–000

Procedia Engineering

www.elsevier.com/locate/procedia

26th International Meshing Roundtable, IMR26, 18-21 September 2017, Barcelona, Spain

Anisotropic mesh adaptation and effects on the conditioning of unstructured finite element solvers

Gabriel Manzinali[a,*], Elie Hachem[a,*], Youssef Mesri[a,*]

[a]MINES Paristech, PSL-Research University, CEMEF-Centre for material forming, 1 rue Claude Daunesse, 06904 Sophia-Antipolis Cedex, France.

Abstract

Over the past decades the growing need for high fidelity flow simulations paved the way to promising techniques such as anisotropic mesh adaptation in finite elements (FE) frameworks. One of the possible drawbacks of anisotropic meshes is their effects on the efficiency of the iterative solver. In this work we investigate the impact of anisotropic adapted meshes on the conditioning of the stiffness matrix, that stems from the FE discretized problem. The results show that, using a simple Jacobi diagonal scaling, we can strongly reduce (or even delete) the gap between isotropic and anisotropic meshes, in term of conditioning.

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Keywords: mesh adaptation; anisotropic mesh; finite element; mass matrix; stiffness matrix; conditioning; preconditioning; diagonal scaling.

1. Introduction

Anisotropic mesh adaptation proved to be a successful technique to solve challenging problems with high accuracy. This technique has been studied and employed with excellent results for many years; however anisotropic meshes might lead to ill-conditioned linear systems and their solution may deteriorate the efficiency of the entire computation. Only few studies focused on these effects on the solver. It is thus important to understand how mesh anisotropy affects the conditioning of linear systems resulting from discretization of PDEs on anisotropic meshes, to asses the effectiveness of this technique.

The linear system is solved using an iterative solver, such as Conjugate Gradient for symmetric problems and GMRES for non-symmetric ones. If the condition number grows without control, the number of iterations needed by solver increases. This results in a loss of efficiency in the simulation.

In this work we apply the anisotropic mesh adaptation process proposed by Mesri et al. [1], driven by a directional error estimator based on the recovery of the Hessian of the finite element solution. The goal is to show how anisotropy affects the iterative solution of the problem, applying the proposed method to numerical examples, namely diffusion and advection-diffusion problems. We investigate the impact of anisotropic adapted meshes on the conditioning of the

* Corresponding authors.
E-mail address: gabriel.manzinali@mines-paristech.fr, elie.hachem@mines-paristech.fr, youssef.mesri@mines-paristech.fr

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stiffness matrix, that stems from the FE discretized problem. Following the approach of Kamenski et al. [2], we show that using a simple Jacobi diagonal scaling as proposed by Bank and Scott [3], we can strongly reduce (or even delete) the effects of mesh irregularity on the conditioning. This indicates that, especially if the mesh concentration is near the boundary, the impact of anisotropy on the performances of the solver is better than what is commonly feared.

2. Mesh adaptation

We present here a reminder of the *a posteriori* error estimator used for the mesh adaptation based on the work of Mesri et al., see [1] and references therein. An approximation of the Hessian matrix is used to estimate the error $e = u - u_h$. The Hessian matrix is recovered from the function u_h using a technique to recover the gradient, based on the approach of Zienkiewicz and Zhu [4]. We use the L^p-norm of the recovered Hessian to build the *a posteriori* error estimator. The re-meshing algorithm requires that the tetrahedrons of the mesh are equilateral in the metric defined by the local metric field $\mathscr{M}$. The metric is defined in $\mathbb{R}^d$ by:

$$\mathscr{M}(P) = \frac{1}{h_1(P)} e_1 \otimes e_1 + \cdots + \frac{1}{h_d(P)} e_d \otimes e_d \tag{1}$$

where $(e_i)_{i=1,d}$ are the eigenvectors of the recovered Hessian $H_R(u_h(P))$ and $h_i(P)$ are the mesh sizes in the e_i directions.

2.1. Anisotropic a posteriori error estimator

Given a mesh discretization $\mathscr{T}_h$ of Ω, we define the anisotropic error estimator of the element $T \in \mathscr{T}_h$ by

$$\bar{\eta}_T = \Big(\int_T \big(\mathscr{H}(u_h(x_T))(x - x_T) \cdot (x - x_T) \big)^p \, dT \Big)^{\frac{1}{p}} \tag{2}$$

where the matrix $\mathscr{H}$ is defined as

$$\mathscr{H} = R \Lambda R^T = |\lambda_1| e_1 \otimes e_1 + \cdots + |\lambda_d| e_d \otimes e_d \tag{3}$$

R is the orthonormal matrix which corresponds to the eigenvectors $(e_i)_{i=1,d}$ of the Hessian matrix. $\Lambda = diag(|\lambda_1|, ..., |\lambda_d|)$ is the diagonal matrix of the absolute values of the eigenvalues of the recovered Hessian.

Substituting (3) in (2) we get,

$$\bar{\eta}_T^p = \int_T \big(\sum_{i=1}^{d} |\lambda_i(x_T)| [e_i(x_T) \cdot (x - x_T)]^2 \big)^p dT \tag{4}$$

The projection of $x - x_T$ on e_i direction is $[e_i \cdot (x - x_T)]^2 = x_i^2 \leq h_i^2$ and then,

$$\bar{\eta}_T^p \leq \int_T \big(\sum_{i=1}^{d} |\lambda_i(x_T)| h_i^2 \big)^p dT \tag{5}$$

If the optimal mesh is the one that is aligned with this solution, the local error in any direction has the same value. In this case the local error in the principal directions of curvature is constant per element.

Using this property we can rewrite the bound as

$$\bar{\eta}_T^p \leq |T| \big(d |\lambda_d(x_T)| h_d^2 \big)^p \tag{6}$$

where $|T|$ is the volume of element T.
Finally we can formulate the local estimator using the proposed upper bound

$$\eta_T \; = \; d |T|^{\frac{1}{p}} |\lambda_d(x_T)| h_d^2 \tag{7}$$

This estimator is used as a functional of a minimization problem. The solution of this problem is the metric used to construct an optimal adapted mesh.

3. Numerical Tests

In this section we present two numerical examples of the application of the adapting proposed method.

For each test we computed the condition number associated to each discretization. First we consider the solution without preconditioning. Secondly we apply a Jacobi diagonal scaling as a preconditioner, following the approach found in [2] and references therein.

We chose a pure diffusion problem and a steady convection diffusion problem, studied by several authors e.g. [5]. The finite element method used is stabilized using a Streamline Upwind Petrov-Galerkin formulation, following the approach developed in [6].

$$\begin{cases} -\nabla \cdot (a\nabla u) + \mathbf{v} \cdot \nabla u &= f, \quad \text{in } \Omega \\ u &= g, \quad \text{in } \Gamma \end{cases} \tag{8}$$

where $\Omega \subset \mathbb{R}^d$ is a bounded polyhedral domain with boundary Γ. For the sake of simplicity we consider Dirichlet boundary conditions. Here a is the diffusion coefficient, $\mathbf{v}(\mathbf{x}) \in \left[W^{1,\infty}(\Omega)\right]$ is the divergence-free velocity field, $f(x) \in L^2(\Omega)$ is a given source term, u_0 is the initial data and g is a given boundary condition.

For both tests we compute the solution comparing isotropic meshes and the anisotropic meshes obtained applying the proposed method. We consider 6 levels of refining from 500 to 100000 elements.

3.1. Diffusion (Poisson)

The first test case is a diffusion problem with continuous solution and regular boundary layers. We consider a squared computational domain $\Omega = (0,1)^2$ and a diffusion coefficient $a = 10^{-2}$. The exact solution is given by $u(x,y) = 4(1 - e^{-x/a} - (1 - e^{-1/a})x)y(1-y)$. The boundary conditions and the source term are determined from the exact solution. In Figure 1 we show the condition number k with respect to the number of elements used. Without

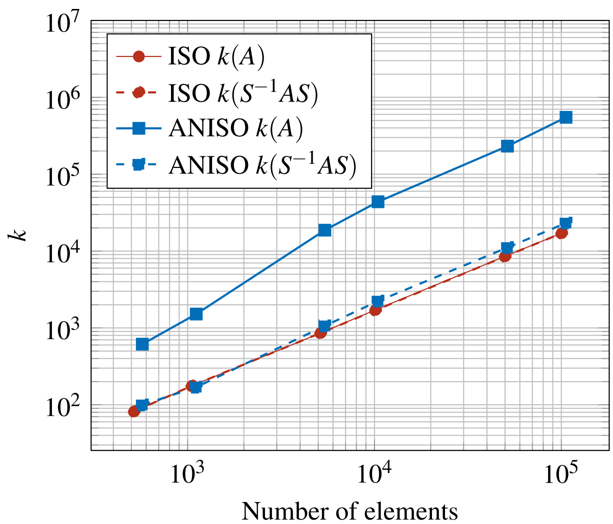
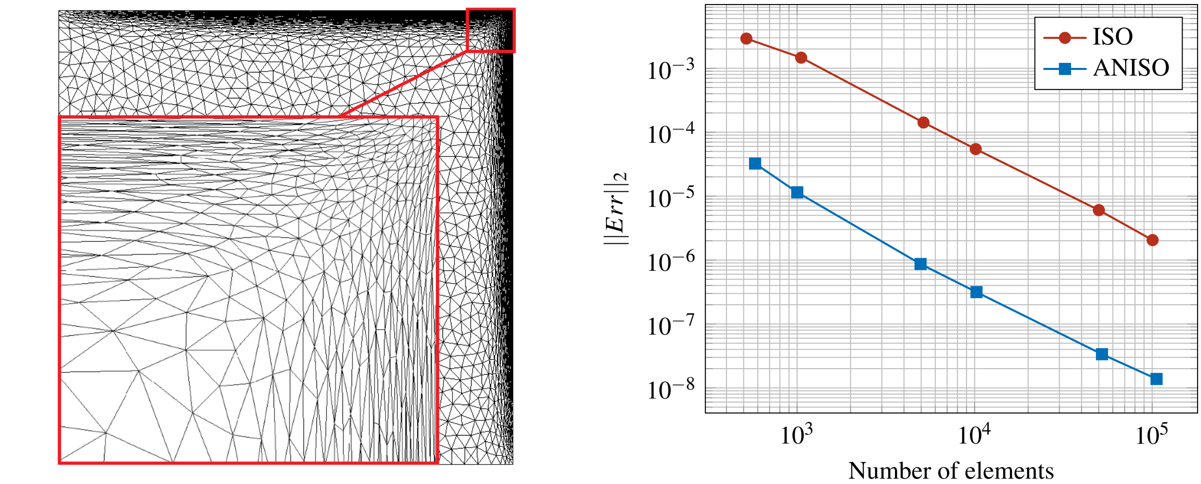


Fig. 1. Condition number vs Number of elements for the diffusion problem. With dashed lines the results using diagonal scaling

any preconditioner it is clear that the anisotropic mesh (blue) is ill-conditioned compared to the isotropic one (red). However, when the Jacobi preconditioner is applied, the conditioning of the two meshes becomes comparable (dashed lines). We can highlight that the diagonalisation has no effect of the uniform meshes.

3.2. Convection-Diffusion

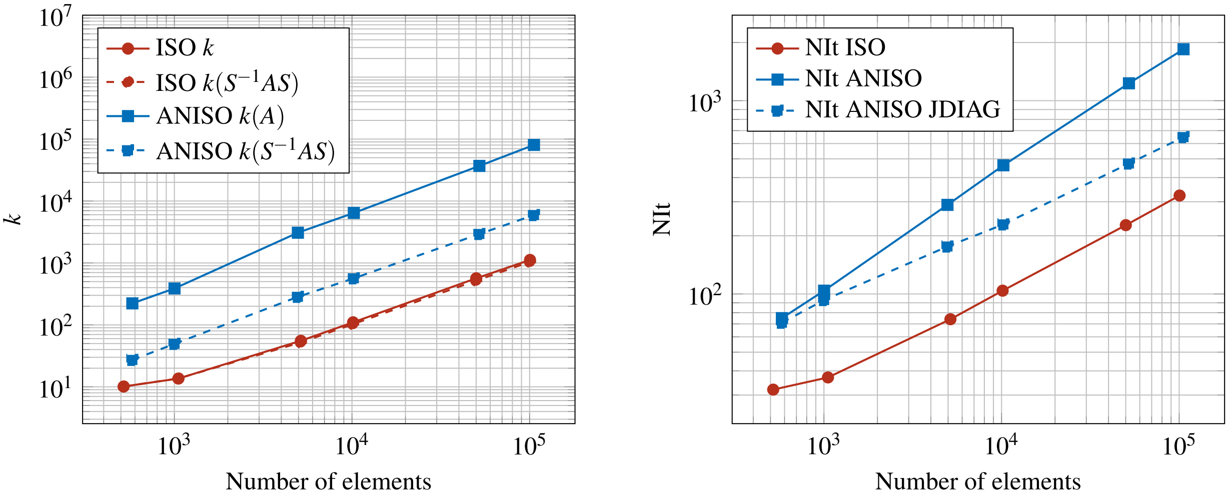
The second test case is a convection-diffusion problem with continuous solution and regular boundary layers. We consider a squared computational domain $\Omega = (0,1)^2$ and a diffusion coefficient $a = 10^{-2}$. The exact solution is given by $u(x,y) = xy\left(1 - e^{\frac{1-x}{a}}\right)\left(1 - e^{\frac{1-y}{a}}\right)$ and develops boundary layers at $x = 1$ and $y = 1$. The boundary conditions and the source term are determined from the exact solution. An example of the adapted anisotropic mesh obtained with 5000 elements is shown in Figure 2(a). The convergence curves in Figure 2(b) show that, using the anisotropic mesh adaptation technique proposed, we can reach higher accuracy with less elements.



(a) Example of one anisotropic adapted mesh (5000 elements)

(b) L^2 norm of the error vs Number of elements

Fig. 2. Mesh example and convergence curves for the Convection-Diffusion problem.



(a) Condition number vs Number of elements

(b) Number of iterations vs Numer of elements

Fig. 3. Results for the Convection-Diffusion problem. With dashed lines the results using diagonal scaling

In Figure 3(a) we show the condition number k with respect to the number of elements used. Without any preconditioner the anisotropic mesh (blue) is very ill-conditioned compared to the isotropic one (red). However, when the Jacobi preconditioner is applied, the conditioning of the anisotropic mesh is highly improved (dashed lines). We can highlight that the diagonalisation has no effect of the uniform meshes. Figure 3(b) shows the impact of conditioning on the number of iterations needed by the linear solver. In this case we consider a GMRES solver with a precision of 10^{-10}. As we can see, the number of iteration needed is reduced by the use of the diagonalisation. Moreover, for the anisotropic mesh this number approaches the value relative to the uniform mesh when we increase the number of elements.

4. Conclusions

In this note we provided an error estimator as a minimization problem in L^P norm. From this minimization we derived the optimal metric used by the anisotropic mesh adaptation tool. We used this technique to study the effects of anisotropic adapted meshes on the conditioning of the stiffness matrix of the FE problem. The results show that these effects can be strongly reduced with a simple Jacobi diagonalisation, resulting in a conditioning that is better than what is commonly feared.

Future work will focus on the study of efficient preconditioning techniques dedicated for the specific case of anisotropic meshes, to furthermore reduce their effect on the performances of the solver. Further investigation will be dedicated to the extension to the case of unsteady Navier-Stokes equations.

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