Research Note
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Generation of Hierarchical Tetrahedral Meshes with High-Order Projections for Efficient Multigrid Solvers

Cao Lu[a], Xinglin Zhao[a], Navamita Ray[b], Xiangmin Jiao[a*]

[a]*Department of Applied Mathematics and Statistics, Stony Brook University, Stony Brook, NY 11794, USA.*
[b]*Mathematics and Computer Science, Argonne National Laboratory, Argonne, IL 60439, USA*

Abstract

We investigate the problem of generating hierarchical unstructured meshes through uniform mesh refinement for efficient solution of PDEs using finite element methods and multigrid solvers. To ensure high accuracy along curved boundaries, we utilize high-order surface reconstruction and projection, without requiring a CAD model. The resulting mesh is then used with a Hybrid Geometric+Algebraic (*HyGA*) multigrid method. We describe the data structure and software requirements, and present numerical results to demonstrate the accuracy and efficiency of multigrid finite-element solver with the proposed techniques.

Keywords: uniform mesh refinement, hierarchical meshes, multigrid methods, high-order projection

1. Introduction

Mesh generation and linear solvers are often the two most expensive steps in the solutions of partial differential equations (PDEs) using finite element methods with unstructured meshes. There are numerous advantages to use hierarchical meshes, including accuracy and efficiency, especially in the context of large-scale parallel computing, as both the number of processors and the mesh resolution increase.

One approach of generating hierarchical meshes is to use uniform mesh refinement (UMR), which is a relatively simple and efficient process. In addition, a key motivation of this work is that UMR can speed up not only mesh generation, but also the linear solvers with multigrid methods. It is well known that geometric multigrid methods (GMG) can theoretically deliver optimal time complexity for solving sparse linear systems from PDE discretizations [1]. However, traditional GMG requires hierarchical meshes with deep hierarchies, which is overly restrictive. Practitioners often utilize algebraic multigrid methods (AMG), which are convenient but far inferior to GMG in efficiency. In our recent work [5], we proposed a hybrid geometric+algebraic multigrid method, or *HyGA*, which combines the efficiency of GMG at finer resolutions with the flexibility and simplicity of AMG at coarser resolutions. Unlike pure GMG, HyGA requires only two or three levels of mesh refinement to deliver comparable efficiency as GMG. In this work, we investigate uniform mesh refinement and its effects on mesh quality, accuracy of finite element methods, and efficiency of multigrid solvers, focusing on tetrahedral meshes.

* Corresponding author. Email: jiao@ams.sunysb.edu.

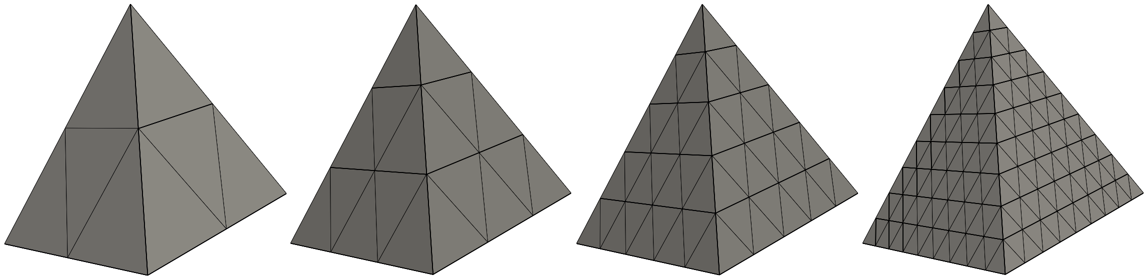


Figure 1: Degree-2, 3, 5, and 9 refinements of a tetrahedron.

While UMR is a relatively simple process, it is by no means trivial. Notable issues include mesh quality, treatment of curved boundaries, multi-level and multi-degree refinement, and data structure and software design. In this work, we show that the overall mesh quality improves during UMR for tetrahedral meshes. To ensure the accuracy of the finite element discretizations, we utilize a high-order surface reconstruction for accurate positioning of new vertices on curved boundaries. We present an extension of the array-based half-facet (AHF) data structure [2] to support hierarchical meshes with multi-level or multi-degree refinements. We also describe two key operations on top of this data structure to enable easy implementation of multigrid methods. Our numerical results demonstrate the accuracy and efficiency of multigrid finite-element solver with the proposed techniques.

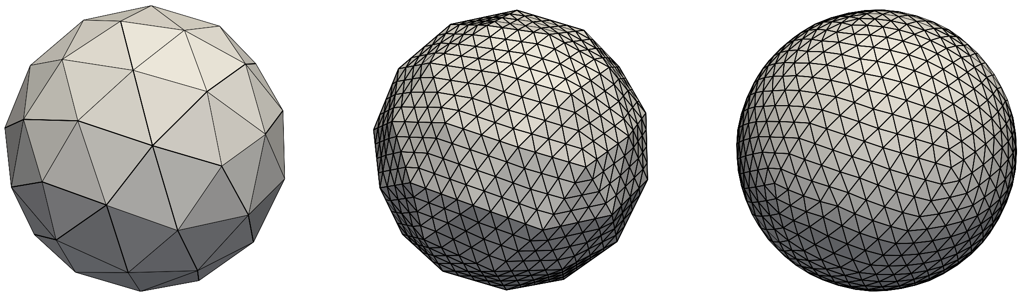
2. Multi-Degree and Multi-Level Hierarchical Meshes

Multi-Degree Refinement. In uniform refinement, we refine a tetrahedron by introducing new vertices in its edges, faces and/or interior. The vertices are positioned analogous to Pascal's tetrahedra (aka Pascal's Pyramids) for trinomial coefficients. We refer to the degree of a Pascal's tetrahedron, i.e., the degree of its corresponding polynomials, as the degree of the refinement. Figure 1 shows examples of degree-2, 3, 5, and 9 refinements of a tetrahedron. In general, a degree-k refinement has $k-1$ mid-edge vertices along each edge. We uniformly subdivide each face into k^2 congruent sub-triangles, and then reconnect the interior vertices to obtain k^3 sub-tetrahedra. Unlike the sub-triangles, these sub-tetrahedra are not congruent, and this raises the question of mesh quality. To deliver good mesh quality, we first obtain a set of congruent sub-tetrahedra and intermediate octahedra, and then subdivide these octahedra into sub-tetrahedra by connecting the shortest diagonals. As shown in [7], this strategy preserves the stability of FEMs. Our experiments show that it also improves the average mesh quality during UMR. To refine the whole mesh, we introduce vertices for all tetrahedra collectively to avoid duplicates, and insert these vertices and sub-tetrahedra into AHF.

Multi-Level Refinement. The uniform refinement described above refines a mesh by one level. The multigrid method requires a sequence of successively refined meshes. One could apply the refinement algorithm repeatedly, but this approach may be inefficient. Instead, our algorithm allows generating a multi-level mesh in a single pass. More specifically, if the finest mesh is degree $K = k_1 k_2 \dots k_L$, where the k_i are prime numbers, we generate a level-$(L + 1)$ mesh by applying a degree-K refinement and then creating the sub-entities in the $L - 1$ intermediate levels.

High-Order Surface Reconstruction. Mesh refinement introduces new vertices. If the boundary is curved, inserting points to the linear edges or faces would compromise the accuracy of the geometry and in turn that of the finite element solver. To address this issue, we use the high-order surface reconstructions as described in [3,4], which can achieve third and even higher order accuracy. Figure 2 shows a comparison of refining a sphere with and without high-order projection. For geometry with sharp features, we detect sharp feature curves and construct 1-D high-order reconstructions for these curves.

Data Structure and Matrix Operations for Multigrid Solvers. To support multigrid solvers, our data structure stores the sub-entities of each level consecutively as well as the parent-children information of adjacent levels with additional arrays in AHF. Instead of exposing all the details of this adjacency information to users, we implement the two key mesh operations in multigrid solvers: *prolongation*, which interpolates data from a coarse mesh to the next finer mesh,



(a) Original Mesh (b) With linear interpolation. (c) With high-order projection.

Figure 2: Linear interpolation leads to inaccurate geometry, which is resolved by high-order surface reconstruction.

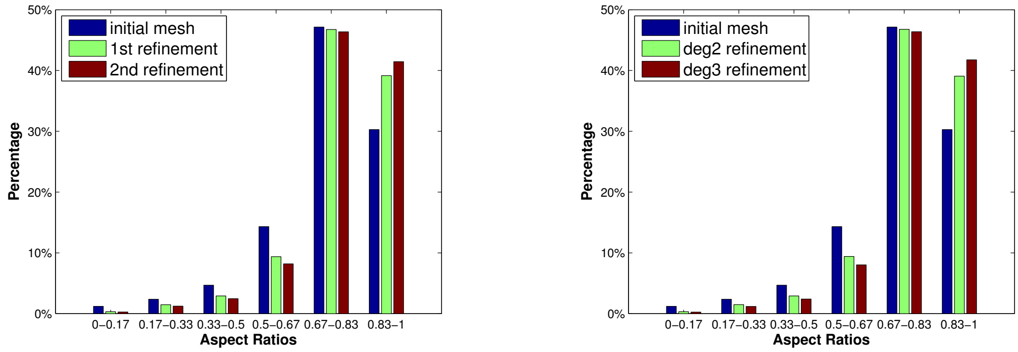


Figure 3: Aspect ratio distributions with two-level degrees 2+2 (left) and degrees 2+3 (right) refinements.

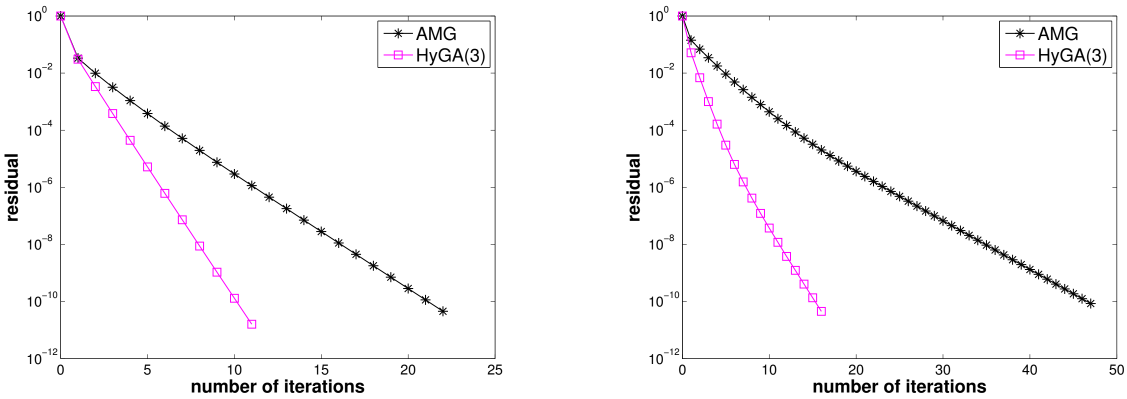
and *restriction*, which transfer data from a fine mesh to the next coarser mesh. Alternatively, we can also output the coefficient matrix for these operations, so that the user can utilize efficient matrix-vector multiplication routines to implement these operations.

3. Numerical Results

We present some numerical results to demonstrate the effectiveness of uniform mesh refinement, in terms of mesh quality, efficiency of multigrid methods, and most importantly, the accuracy of finite element methods. We use a cylinder as our test geometry, as it includes both curved regions and sharp features.

Mesh Quality Under Uniform Mesh Refinement. For tetrahedral meshes, uniform mesh refinement does not produce congruent sub-tetrahedra. It is interesting to note that the average mesh quality actually improves during UMR with the shortest-diagonal approach. To demonstrate it, we generate an initial coarse tetrahedral mesh using TetGen [6], with approximately 5K vertices and 2.5K tetrahedra. We uniformly refine twice with two strategies. First, we refine it using two degree-2 refinements. The resulting finest mesh has about 280K vertices and 1.6M tetrahedra. Second, we refine it using one degree-2 followed by a degree-3 refinement. We measure the mesh quality using the aspect ratio, computed as three times the inscribed radius (IR) divided by the circumsphere radius (CR), i.e., $3 \times \text{IR}/\text{CR}$. Larger aspect ratio corresponds to better element quality. The two graphs in Figure 3 show the distributions of the aspect ratios on each level for the two strategies, respectively. In both cases, the overall mesh quality improved, and degree-3 refinement delivers similar and even slightly better quality improvement because there are more intermediate octahedra in degree-3 refinement.

Efficiency of Multigrid Methods. To demonstrate the effectiveness of multigrid methods under UMR, we solve the Poisson equations using finite element methods over the three-level meshes described above. We refer to the solver as



(a) Convergence with degrees-2+2 refinement.

(b) Convergence with degrees-2+3 refinement.

Figure 4: Convergence of multigrid methods with two-level degrees 2+2 (left) and degrees 2+3 (right) refinements.

Table 1: Errors in finite element solutions with linearly interpolated boundary and high-order reconstructions.

Level	Linearly interpolated boundary		High-order reconstructed boundary	
	global error	boundary error	global error	boundary error
2nd	$4.7e-1$	$4.2e-1$	$6.6e-1$	$6.2e-1$
3rd	$3.8e-1$	$3.8e-1$	$1.5e-1$	$1.1e-1$
4th	$4.4e-1$	$4.4e-1$	$5.0e-2$	$5.0e-2$

HyGA(3), and compare it against classical AMG. The two plots in Figure 4 show the convergence results for the two meshes, respectively. It can be seen that HyGA converges twice as fast as AMG for degree-2 refinements, and about three times as fast in the presence of a degree-3 refinement. This result is not surprising, because as we have shown in [5], HyGA with even only two or three levels of mesh hierarchy can deliver similar performance as a pure GMG when a pure GMG is applicable. However, a pure GMG is not possible here, because there is an insufficient number of mesh levels. HyGA avoids the need of a deep mesh hierarchy for a pure GMG, and at the same time substantially reduces the computational cost of a pure AMG.

Accuracy of Finite Element Solutions. The ultimate judgement of the quality of a mesh and linear solver is the accuracy of computational results. One key aspect that affects the accuracy is the high-order projection of the boundary vertices, because simple linear interpolation for the newly inserted boundary vertices may destroy the convergence of the finite element methods. To illustrate this effect, we start with a coarse initial mesh, shown in Figure 5(a). We refine the mesh with degree-2 refinements, and compare the results with and without high-order projections. Figure 5(b) shows the resulting mesh with linearly interpolated vertices, and Figure 5(c) shows the mesh with projection onto the high-order reconstructed surface. The latter mesh is visually much smoother and more accurate. We then solve the Poisson equations with Dirichlet boundary condition on the two refined meshes, with $e^{x^2+y^2+z^2}$ as the analytical solution. The boundary conditions are determined based on the closest points on the cylinder. Table 1 shows the errors of the finite element solution in infinity norm under different levels of UMR, where the "global error" considers all the interior vertices and the "boundary error" only the vertices directly connected to the boundary. It is clear that with linear interpolation of boundary points, the convergence of finite element methods is totally lost, and the largest errors occur near the boundary. In contrast, with high-order reconstruction, the solution preserves the second-order convergence of the finite element method. Therefore, uniform mesh refinement coupled with high-order reconstruction is an effective strategy for its simplicity, accuracy, and high efficiency of the linear solver.

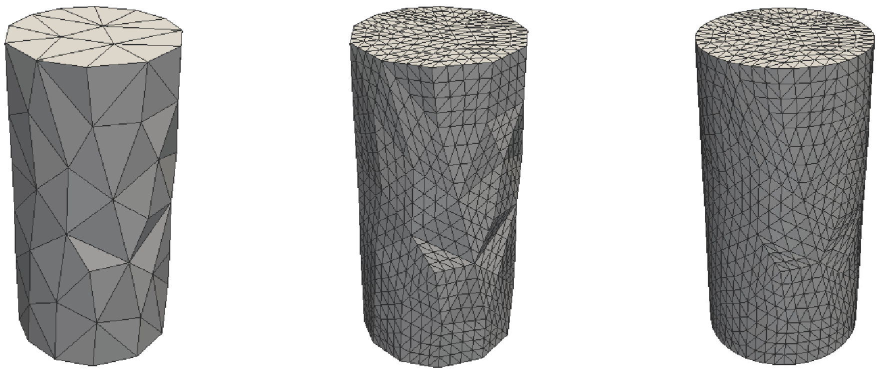


Figure 5: Uniform refinement of a mesh (left) without (middle) and with (right) high-order surface reconstruction.

4. Conclusion and Discussion

In this paper, we presented a method for generating a hierarchical tetrahedral mesh for a hybrid multigrid solver (HyGA) for PDEs. We utilize a high-order surface reconstruction strategy for accurate treatment of curved boundaries without requiring a CAD model. Our numerical results demonstrate that this strategy significantly improves the accuracy of the finite-element solvers. The hierarchical meshes enable HyGA to solve linear systems efficiently, which are often the most time-consuming part in solving PDEs. This preliminary work enables additional opportunities for efficient and parallel mesh generation for large-scale parallel computers, which we are pursuing and plan to report in future publications.

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