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Generation of polyhedral Delaunay meshes

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Abstract

A polyhedral mesh fulfills the Delaunay condition if the vertices of each polyhedron are co-spherical and each polyhedron circumsphere is point-free. If Delaunay tessellations are used together with the finite volume method, it is not necessary to partition each polyhedron into tetrahedra; co-spherical elements can be used as final elements. This paper presents a mixed-element mesh generator based on the modified octree approach that has been adapted to generate polyhedral Delaunay meshes. The main difference with its predecessor is to include a new algorithm to compute Delaunay tessellations for each 1-irregular cuboids (cuboids with at most one Steiner point on their edges) that minimize the number of mesh elements. In particular, we show that when Steiner points are located at edge midpoints, 24 different co-spherical elements can appear while tessellating 1-irregular cubes. By inserting internal faces and edges to these new elements, this number can be reduced to 13. When 1-irregular cuboids with aspect ratio equal to $\sqrt{2}$ are tessellated, 10 co-spherical elements are required. If 1-irregular cuboids have aspect ratio between 1 and $\sqrt{2}$, all the tessellations are adequate for the finite volume method. The proposed algorithm can be applied to any point set to compute the Delaunay tessellation inside the convex hull of the point set. Simple polyhedral Delaunay meshes generated by using the adapted mesh generator are shown.

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Keywords: Delaunay tessellations; Polyhedral meshes; Octrees.

1. Introduction

Scientific and engineering problems are usually modeled by a set of partial differential equations and the solution to these partial differential equations is calculated through the use of numerical methods. In order to get good results, the object being modeled (domain) must be discretized in a proper way respecting the requirements imposed by the used numerical method. The discretization (mesh) is usually composed of simple cells (basic elements) that must represent the domain in the best possible way. In particular, we are interested in meshes for the finite volume method [1] which are formed by polygons (in a 2D domain) or polyhedra (in a 3D domain), that satisfy the Delaunay condition: the circumcircle in 2D, or circumsphere in 3D, of each element does not contain any other mesh point in its interior [2]. A Delaunay mesh is required because we use its dual structure, the Voronoi diagram, to model the control volumes in order to compute an approximated solution [3–6]. The basic elements used so far are triangles and quadrilaterals

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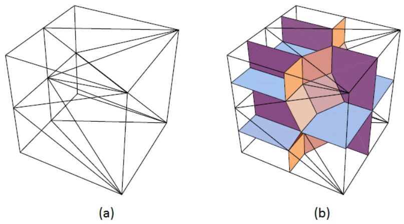


Fig. 1. (a) Mixed mesh of a 1-irregular cuboid that satisfies the Delaunay condition, (b) the same mixed mesh and its associated Voronoi diagram.

in 2D, and tetrahedra, cuboids, prisms and pyramids in 3D. Meshes composed of different elements types are called mixed element meshes [4] or polyhedral meshes [7]. The advantage of using a mixed mesh in comparison with a tetrahedral mesh is that the use of cuboids allows to represent very thin layers with fewer elements and, in general, the use of different element types reduce the amount of edges, faces and elements in the final mesh. On the other hand, a disadvantage is that the equations must be discretized using different elements or to include the possibility to manage general polyhedra [7].

Octree based approaches [8,9] naturally produces co-spherical point sets. A mixed mesh satisfying the Delaunay condition can include all produced co-spherical elements as shown in Figure 1. The final elements in this example are five pyramids and four tetrahedra.

This paper presents an algorithm to generate polyhedral Delaunay meshes from any point set. This algorithm has been integrated into a previous mesh generator used to generate mixed element meshes and currently, we can generate simple polyhedral Delaunay meshes that fulfil the point density requirements specified by an user. We have identify all co-spherical elements that can appear while tessellating 1-irregular cubes generated by using a bisection based approach. Statistics associated with particular tessellations are presented such as the frequency each co-spherical element is used. We have also identify when the algorithm can be used to tessellate locally 1-irregular configurations, maintaining the global Delaunay condition, without adding extra vertices.

In this paper, we have focused our work on the analysis of the tessellations of 1-irregular cuboids because this element is the one that more frequently appears when meshes are generated by a modified octree approach. A theoretical study on the number of different 1-irregular cuboid configurations that can appear either by using a bisection or an intersection based approach was published in [10].

This paper is organized as follows: Section 2 describes basic concepts and the previous mesh generator based on modified octrees. Section 3 presents briefly the developed algorithm to compute Delaunay tessellations. Section 4 shows the application of the tessellation algorithm to 1-irregular cuboids generated by a bisection based approach. Section 5 shows some simple polyhedral meshes and Section 6 includes our conclusions.

2. Basic concepts

In this section we define first the concept of Delaunay tessellations, and then we introduce the mesh generator based on modified octrees.

Definition 1. *A tessellation T of a set of points S is a* Delaunay tessellation *if there exists a point-free circumsphere for each polyhedral element.*

We use the term *Delaunay tessellation* and not *Delaunay triangulation* [2,11–13] because our meshes include element types other than tetrahedra if their vertices are co-spherical. The best known of these elements are cuboids and some kinds of prisms and pyramids. Delaunay tessellations are very useful in control volume methods that use the Voronoi region as integration volume. Co-spherical configurations (elements) that satisfy the Delaunay condition are not required to be tessellated into smaller elements because the numerical method only needs the Delaunay edges with associated Voronoi edges in 2D (faces in 3D) whose length (area) is not equal to 0.

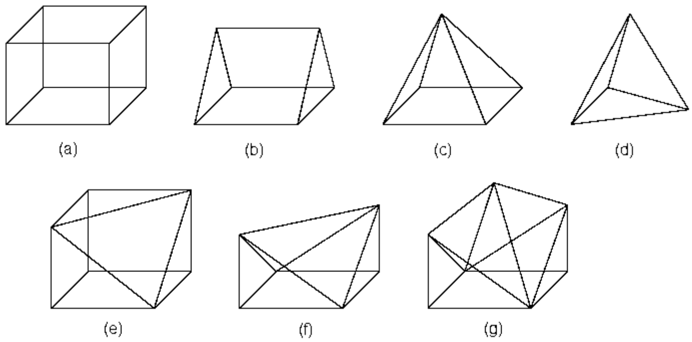


Fig. 2. The seven final elements of the Ω Mesh Generator: (a) Cuboid, (b) Triangular Prism, (c) Quadrilateral Pyramid, (d) Tetrahedron, (e) Tetrahedron Complement, (f) Deformed Prism, and (g) Deformed Tetrahedron Complement.

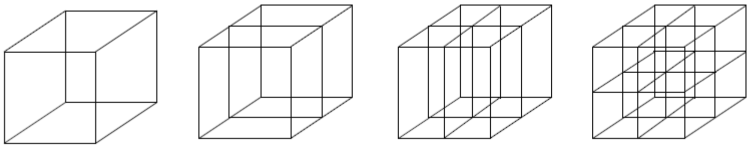


Fig. 3. Cuboid and its splits into two, four and eight cuboids using a bisection based approach.

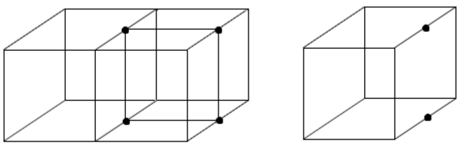


Fig. 4. The bisection-refined cuboid at the left produces an 1-irregular element like the cuboid at the right.

2.1. Mesh generation algorithm

We have developed a mixed element mesh generator [14] based on an extension of octrees [8,9]. Our approach starts enclosing the domain in the smallest bounding box (cuboid). Second, this cuboid is continuously refined, at any edge position, by using the geometry information of the domain. That is why this refinement is called intersection based approach. Once this step finishes, an initial non-conforming mesh composed of tetrahedra, pyramids, prisms, and cuboids is generated that fits the domain geometry. Third, these elements are further refined by bisection, as far as possible, until the density requirements are fulfilled. In particular, cuboids can be refined into two, four or eight smaller cuboids as shown in Figure 3. Fourth, the mesh is done 1-irregular by allowing only one Steiner point on each edge. An example of a 1-irregular element is shown in Figure 4. In order to tessellate 1-irregular elements, mainly a template based approach is used but only the most frequently used patterns are available. Then, if a pattern is not available or the element can not be properly tessellated for the finite volume method, new Steiner points are inserted until all 1-irregular elements can be properly tessellated. The current set of seven final elements is shown in Figure 2. One of the main drawbacks of this mesh generator is that if 1-irregular elements can not be properly tessellated, several points must be added. Since any polyhedral Delaunay tessellation could be used, we have designed and implemented an algorithm that for any point set, computes the Delaunay tessellation inside its convex hull. This algorithm is described in the next section.

3. Algorithm for generating Delaunay tessellations of any point set

Let P be a point set in 3D. The Delaunay tessellation of P inside its convex hull can be computed following the next main steps:

1. Build a Delaunay tetrahedral mesh for P. We do it by using QHull [15][1].
2. Join tetrahedra to form the largest possible co-spherical elements.
3. Build each co-spherical polyhedron from the respective set of tetrahedra: the faces of the polyhedron are the faces of the convex hull of the tetrahedron set.

Qhull divides co-spherical point configurations into a set of tetrahedra by adding an artificial point that is not part of the input. Then, the algorithm uses this fact to recognize the tetrahedra that form a co-spherical polyhedron and later to recognize which element is.

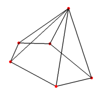
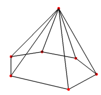
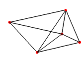
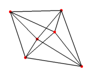
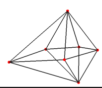
4. Applying the tessellation algorithm to 1-irregular cuboids

In this section, we present the polyhedra obtained by applying the algorithm described in the previous section to the 4096 (2^{12}) 1-irregular configurations that can be generated while refining a cuboid with a bisection based approach [10]. First, the new co-spherical elements are shown. Then, their impact in all the tessellations is analyzed and finally, we show for which cuboid aspect ratio, the tessellation algorithm can be used to tessellate locally 1-irregular cuboids keeping the global Delaunay condition.

4.1. New co-spherical elements

We have identified 17 new co-spherical polyhedra in the tessellations of 1-irregular cubes in addition to the seven original elements shown in Figure 2. A description of each one can be found in Table 1.

Table 1. Description of the new co-spherical elements that appear while tessellating 1-irregular cubes

Element	Vertices	Edges	Faces	Example
Pentagonal Pyramid	6	10	6	
Hexagonal Pyramid	7	12	7	
Triangular Bipyramid	5	9	6	
Quadrilateral Bipyramid	6	12	8	
Pentagonal Bipyramid	7	15	10	

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[1] http://www.qhull.org

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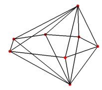
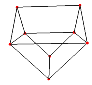
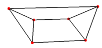
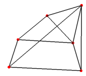
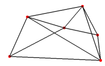
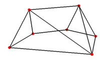
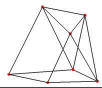
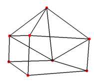
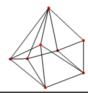
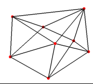
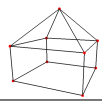
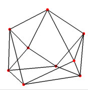
Element	Vertices	Edges	Faces	Example
Hexagonal Bipyramid	8	18	12	
Triangular Biprism	8	14	8	
Generic #1	6	9	5	
Generic #2	6	10	6	
Generic #3	6	11	7	
Generic #4	7	12	7	
Generic #5	7	13	8	
Generic #6	8	15	9	
Generic #7	8	16	10	
Generic #8	8	17	11	
Generic #9	9	16	9	
Generic #10	9	18	11	

4.2. Element analysis

Since there are 17 new co-spherical elements, the natural question is if this number can be reduced without adding diagonals in the 1-irregular cuboid rectangular faces. In fact, our mixed element mesh generator requires to tessellate 1-irregular cuboids without adding diagonals on its rectangular faces when it uses a pattern-wise approach. In the following, we analyze the number of co-spherical elements under three different criteria:

- **Finding the optimal tessellation**: An optimal tessellation contains the lowest amount of final elements. This is reached by maximizing the number of elements with different shape. The number of co-spherical elements that can be used is 24.

- **Minimizing the number of different co-spherical elements by adding only internal faces**: Under this criterion we can reduce the number of different final elements by adding only internal faces. Examining the set of new elements in Table 1, we see that the bipyramids and the biprisms are naturally divisible into two elements, and so are the generic #5 (separable into a prism and a quadrilateral pyramid), generic #8 (separable into a prism and two quadrilateral pyramids) and generic #9 (separable into a cuboid and quadrilateral pyramid), among others. An example of this type of separation is shown in Figure 5. The total number of co-spherical elements needed to tessellate the 4096 configurations is now reduced to 16.

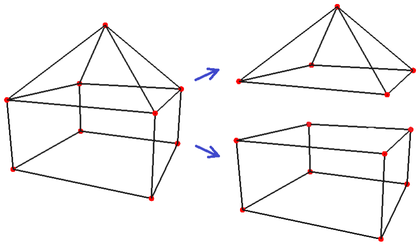


Fig. 5. Generic #9 element and its separation into two different elements.

- **Minimizing the number of different co-spherical elements by adding internal edges and faces**: This extends the second criterion by adding the condition that it is possible to add extra edges only if they are inside the new elements. The reason for only allowing internal edges is that adding external edges could change the partition of one of the rectangular faces of the original cuboid. Under this criterion, the elements that are separable are generic #3 (one inner edge produces two tetrahedra and one quadrilateral pyramid), generic #6 (one inner edge produces two tetrahedra and a tetrahedron complement) and generic #7 (two inner edges produce two tetrahedra, a quadrilateral pyramid and a deformed prism). An example of this type of separation is shown in Figure 6. The total number of co-spherical elements needed to tessellate the 4096 configurations is reduced to 13.

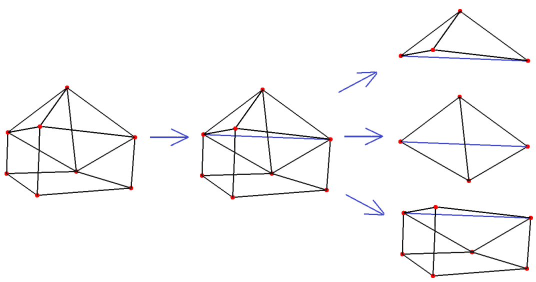


Fig. 6. Generic #6 element and its separation into three different elements by adding an extra inner edge shown in blue.

4.3. Evaluating the impact of each new element

In this section we study how many times each co-spherical element appears in the tessellation of a 1-irregular cuboid. For this study, we have run our program for 1-irregular cuboids with three different aspect ratio: 1, 4, and $\sqrt{2}$.

- **Test case A**. Aspect ratio equal to 1 ($a = b = c$): The 1-irregular cube appears naturally on the standard octree and this method is used by most mesh generators based on octrees.
- **Test case B**. Aspect ratio equal to 4 ($4a = 2b = c$): This represents a typical cuboid to model thin zones.
- **Test case C**. Aspect ratio equal to $\sqrt{2}$ ($a\sqrt{2} = b = c$): It was shown in [3] that some 1-irregular cuboid within these proportions can be tessellated without problems for the finite volume method.

4.3.1. Running the test case A

Table 2 shows the frequency in which appear each one of the 24 co-spherical elements in the tessellations of 1-irregular cubes.

Table 2. Frequency of the co-spherical elements on 1-irregular cube tessellations

Element	Frequency	Element	Frequency
Cuboid	195	Hexagonal Bipyramid	36
Tetrahedron	18,450	Triangular Biprism	6
Quadrilateral Pyramid	11,718	Generic #1	12
Triangular Prism	3,720	Generic #2	96
Tetrahedron Complement	992	Generic #3	48
Deformed Prism	396	Generic #4	48
Deformed Tetrahedron Complement	144	Generic #5	120
Pentagonal Pyramid	384	Generic #6	24
Hexagonal Pyramid	56	Generic #7	48
Triangular Bipyramid	240	Generic #8	48
Quadrilateral Bipyramid	272	Generic #9	6
Pentagonal Bipyramid	192	Generic #10	8
		Total	**37,259**

From Table 2, we observe that the most used elements correspond to tetrahedra and quadrilateral pyramids (~49.5% and ~31.5% of the total elements, respectively). Moreover, the set of seven initial co-spherical elements represents ~95.6% of the total.

4.3.2. Running the test Case B

When the aspect ratio of the cuboid is changed to 4, only 6 different co-spherical elements appear and their frequencies are shown in Table 3.

Table 3. Frequency of the co-spherical elements in the tessellations of 1-irregular cuboids with aspect ratio 4

Element	Frequency	Element	Frequency
Cuboid	103	Triangular Prism	3,120
Tetrahedron	29,118	Tetrahedron Complement	536
Quadrilateral Pyramid	12,620	Deformed Prism	84
		Total	**45,581**

From Table 3, we observe that tetrahedra and quadrilateral pyramids are the most used elements, comprising more than 90% of the total of the elements (~63.9% of tetrahedra and ~27.7% quadrilateral pyramids). Note that these elements can not be divided into simpler ones without adding diagonals on its quadrilateral faces.

4.3.3. Running the test Case C

When the aspect ratio is equal to $\sqrt{2}$, only 10 different final co-spherical elements appear whose frequencies are distributed as follows:

Table 4. Frequency of the final elements on optimal tessellations on 1-irregular cuboids with aspect ratio $\sqrt{2}$

Element	Frequency	Element	Frequency
Cuboid	199	Deformed Prism	284
Tetrahedron	25,252	Triangular Bipyramid	128
Quadrilateral Pyramid	12,300	Quadrilateral Bipyramid	52
Triangular Prism	3,780	Generic #2	16
Tetrahedron Complement	1,008	Generic #5	16
		Total	**43,035**

The most used elements correspond to tetrahedra and quadrilateral pyramids (~58.7% and ~28.6% of the total number of elements, respectively). Moreover, the set of initial seven co-spherical elements represents a ~99.5% of the total.

4.4. Tessellations and the finite volume method

We have also examined whether the generated tessellations meet the requirements for their use in the context of the finite volume method. The requirement is that the circumcenter of each final element is contained within the initial 1-irregular cuboid. This requirement is strong but it allows our mesh generator to apply locally the tessellation algorithm to each 1-irregular cuboid while keeping the global Delaunay condition.
The evaluation of each tessellation is performed on the same test cases discussed in Section 4.2. The results are shown in Table 5. We observe that the circumcenters of all elements are inside the initial cuboid for all the configurations in the test cases A and C. This means that all 1-irregular configurations could be properly tessellated if the aspect ratio of the elements is less or equal to $\sqrt{2}$. If 1-irregular cuboids has an aspect ratio equal to 4, only 132 1-irregular cuboids fit the circumcenter requirement.

Table 5. Number of configurations that fit the circumcenter requirement

Test cases	Number of proper configurations
Test Case A (Aspect ratio equal to 1)	4096
Test Case B (Aspect ratio equal to 4)	132
Test Case C (Aspect ratio equal to $\sqrt{2}$)	4096

5. Examples of simple polyhedral Delaunay meshes

Figure 7 shows a mixed-element mesh generated by the previous mesh generator and Figure 8 shows the polyhedral mesh generated by the current mesh generator that uses the algorithm described in section 3. Table 6 shows the number of polyhedral elements with the same number of faces generated by each one. It can be seen that as expected, when more polyhedral elements can be used, the mesh can satisfy the density requirements with less elements.

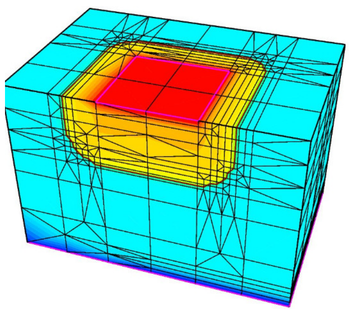


Fig. 7. Mixed element mesh generated by the previous mesh generator

Table 6. Number of elements generated by each mesh generator

	4 faces	5 faces	6 faces	7 faces	8 faces	Total
Previous	478	1163	692	94	0	2427
Current	425	1090	635	100	20	2270

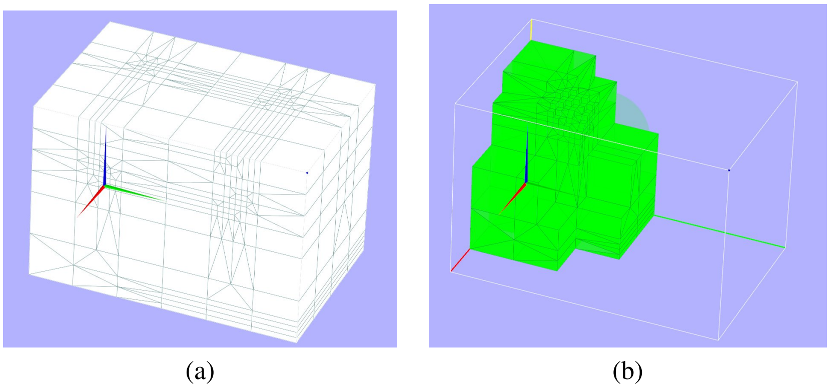


Fig. 8. Polyhedral tessellation of a simple Diode represented by a cuboid (a) external view and (b) internal view

6. Conclusions

We have adapted a mixed-element Delaunay mesh generator that used seven co-spherical elements to one that uses any co-spherical polyhedron if the whole mesh fulfills the Delaunay condition. Currently, we can generate polyhedral Delaunay meshes for simple geometries but we working in order to manage non-convex geometries.

We have proposed an algorithm to compute a Delaunay tessellation of any point set. We have applied it to locally tessellate 1-irregular cuboids but it can also be used without any modification to tessellate a convex region that includes several 1-irregular elements. The circumcenter requirement described in Section 4.4 is only necessary for 1-irregular elements that are located at the boundary or at a material interface, and not among 1-irregular elements. We are also extending the mesh generator to handle this case.

We have identified 24 co-spherical elements while tessellating 1-irregular cubes generated by a bisection based approach. We have experimentally noticed that in the tessellation of 1-irregular cubes (aspect ratio equal to 1) more co-spherical elements appear than in the tessellation of 1-irregular cuboids with larger aspect ratio. When we increase the cuboid aspect ratio a subset of these co-spherical elements is required and no new co-spherical element appears.

We have studied the tessellations of 1-irregular cuboids generated by a bisection based approach with three different aspect ratios: 1, $\sqrt{2}$, and 4. The results can be summarized as follows:

- All the tessellations for 1-irregular cubes and 1-irregular cuboids with aspect ratio from 1 to $\sqrt{2}$ are adequate for the finite volume method. Then if 1-irregular cuboids have this aspect ratio, the algorithm can be used to locally tessellate these elements and the global Delaunay condition will be kept.
- The number of different co-spherical elements while tessellating 1-irregular cubes can be reduced from 24 to 16 by adding internal faces and to 13 by adding internal faces and edges. While tessellating 1-irregular cuboids with aspect ratio equal to $\sqrt{2}$, the required elements are reduced from 10 to 7 if we allow the insertion of internal faces. While tessellating 1-irregular cuboids with aspect ratio equal to 4 only 6 co-spherical elements are used.

We have made the study under the assumption that all the 1-irregular configurations appear in the same rate, but this is for sure not true. While generating a mesh with our mesh generator, there are some configurations that appear more frequently than others. This fact could mean that some co-spherical elements belonging to the tessellation of few 1-irregular cuboids, could have a greater impact than the one we have computed if these few configurations appear very frequently while generating a mesh. A complete study should consider also this case.

The algorithm presented here has been very useful for our mesh generator based on modified octrees, but we think that it can also be useful to adapt other mesh generators based on octrees to generate polyhedral meshes.

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