On shape-regularity of polyhedral meshes for solving PDEs

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1 Introduction

Polyhedral and generalized polyhedral cells appear naturally in reservoir models simulating thinning or tapering out (”pinching out”) of geological layers. The pinch-outs are modeled with mixed types of mesh cells including pentahedrons, prisms and tetrahedrons which are obtained by collapsing pairs of vertices in a structured hexahedral or prismatic mesh. The polyhedral meshes are used actively in a number of hydrodynamics applications [3]. Other sources of polyhedral meshes are the adaptive mesh refinement methods. A locally refined mesh may be considered as the conformal polyhedral mesh with degenerate cells (for instance, when the angle between two neighboring faces in a cell is zero). Usage of polyhedral cells allows us to avoid superfluous mesh refinement.

In contract to Voronoi meshes (see e.g., [5] and references therein), arbitrary polyhedral meshes provide greater flexibility for meshing complex domains. For instance, badly shaped tetrahedra such as slivers can be merged with their neighbors forming shape-regular polyhedra.

Extension of modern discretization methods to polyhedral cells having complex shapes is relatively easy [6, 2]. Indeed, calculations in these methods are performed on the surface of a polyhedral cell, which is a lower-dimensional manifold and hence is easier to treat numerically. These methods impose weak restrictions on shapes of admissible polyhedral cells (see, Fig. 2), and allows us to build optimal-order discretization schemes for a large variety of PDEs on almost arbitrary meshes.

Overall, non-Voronoi polyhedral meshes are quite competitive and in some application areas are preferable to simplicial meshes [4]. In this note, we summarize various existing shape regularity requirements that have to be respected by the developers of polyhedral mesh generators. This summary has been written in a hope to stimulate more research on polyhedral meshes.

2 Shape-regular polyhedral meshes

A polyhedron P is usually defined as a closed domain in three dimensions with flat faces and straight edges. Analysis of discretization schemes is typically conducted on a sequence of conformal polyhedral meshes $\{\Omega_h\}_h$ where h is the diameter of the largest cell in Ω_h and $h \to 0$. A polyhedral mesh is called conformal if intersection of any two distinct polyhedra P_1 and P_2 is either empty, or a few mesh points, or a few mesh edges, or a few mesh faces (two adjacent cells may share more than one edge or more than one face).

Let $|\mathcal{O}|$ denote the Euclidean measure of a mesh object $\mathcal{O}$ and $h_{\mathcal{O}}$ be its diameter. Let $\mathcal{N}_\star$, $\rho_\star$, $\gamma_\star$ and $\tau_\star$ denote various mesh independent constants that are explained below. A polyhedral mesh should satisfy some minimum shape-regularity conditions in order to guarantee optimal error estimates in PDE solvers that depend only on the above star-constants.

(**M1**) Every cell P has at most $\mathcal{N}_\star$ faces and each face f has at most $\mathcal{N}_\star$ edges.

(**M2**) For every polyhedron P with faces f and edges e, we have

$$\rho_\star\, h_{\mathsf{P}}^3 \leq |\mathsf{P}|, \quad \rho_\star\, h_{\mathsf{P}}^2 \leq |\mathsf{f}|, \quad \rho_\star\, h_{\mathsf{P}} \leq |\mathsf{e}|. \tag{1}$$

(**M3**) For each face f, there exists a point $\mathbf{x}_{\mathsf{f}} \in \mathsf{f}$ such that f is star-shaped with respect to every point in the disk of radius $\gamma_\star h_{\mathsf{f}}$ centered at $\mathbf{x}_{\mathsf{f}}$ as illustrated in Fig. 1.

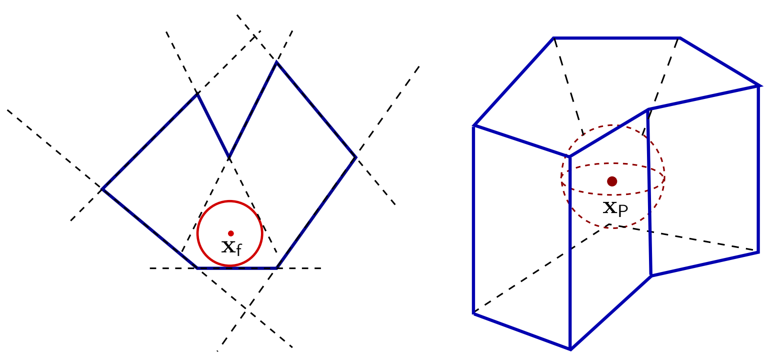

Fig. 1. Left: a feasible set and a polygonal face f star-shaped with respect to the disk centered at $\mathbf{x}_{\mathsf{f}}$. Right: a non-convex polyhedral cell P star-shaped with respect to the sphere centered at $\mathbf{x}_{\mathsf{P}}$.

(**M4**) For each cell P, there exists a point $\mathbf{x}_{\mathsf{P}}$ such that P is star-shaped with respect to every point in the sphere of radius $\gamma_\star\, h_{\mathsf{P}}$ centered at $\mathbf{x}_{\mathsf{P}}$.

(**M5**) For every $\mathsf{P} \in \Omega_h$, and for every $\mathsf{f} \in \mathsf{P}$, there exists a pyramid Q_{f} contained in P such that its base equals to f, its height equals to $\gamma_\star\, h_{\mathsf{P}}$ and the projection of its vertex onto f is $\mathbf{x}_{\mathsf{f}}$.

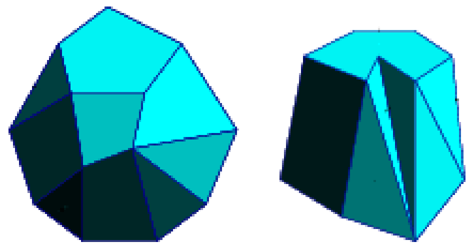

Fig. 2. Shape-regular convex (left) and degenerate non-convex (right) polyhedra.

Two examples of shape-regular polyhedra are shown in Fig. 2. The conditions (**M1**)-(**M5**) are sufficient to develop an *a priori* error analysis of various discretization schemes. We recall only two results underpinning this error analysis. The first one is the Agmon inequality that uses (**M5**) and allows us to bound traces of functions. It states that for any function q in the Sobolev space $H^1(\mathsf{P})$, we have:

$$\sum_{\mathsf{f}\in\partial\mathsf{P}} \|q\|^2_{L^2(\mathsf{f})} \leq C\left(h_{\mathsf{P}}^{-1}\|q\|^2_{L^2(\mathsf{P})} + h_{\mathsf{P}}|q|^2_{H^1(\mathsf{P})}\right). \tag{2}$$

The second one is the following approximation result crucial for proving *a priori* error estimates. Let m be an integer. Then, for any function $q \in H^{s+1}(\mathsf{P})$ with $0 \leq s \leq m$, there exists a polynomial q^m of order at most m such that

$$\|q - q^m\|_{L^2(\mathsf{P})} + \sum_{k=1}^{s} h_{\mathsf{P}}^k |q - q^m|_{H^k(\mathsf{P})} \leq C h_{\mathsf{P}}^{s+1} |q|_{H^{s+1}(\mathsf{P})}. \tag{3}$$

For error analysis of problems appearing in fluid flows and structural mechanics that is based on conditions (**M1**)-(**M5**), we refer to [6] and the extensive list of references therein.

3 An equivalent set of sufficient conditions

The above shape-regularity conditions are satisfied by a wide class of polyhedral meshes that may include non-convex or degenerate cells. Here, we give a shorter set of equivalent conditions that was inspired by a finite element analysis on simplicial meshes [1].

(**A1**) Every polyhedron $\mathsf{P} \in \Omega_h$ admits a conformal decomposition T_h that is made of less than $\mathcal{N}_\star$ tetrahedra and includes all vertices of P.

(**A2**) Each tetrahedron $\mathsf{T} \in \mathsf{T}_h$ is shape-regular: the ratio of radius r_{T} of the inscribed sphere to diameter h_{T} is bounded from below:

$$r_{\mathsf{T}} \geq \rho_\star h_{\mathsf{T}}.$$

(**A3**) Each cell P (resp., each face f) is star-shaped with respect to the centroid of a tetrahedron $\mathsf{T} \in \mathsf{T}_h$ (resp., a triangle in the surface mesh $\mathsf{T}_h|_{\mathsf{f}}$).

We stress that only existence of a tetrahedral partition T_h is required, a fact that can be easily verified in most cases. Moreover, these partitions are not required to match across cell boundaries.

4 Shape-regular generalized polyhedral meshes

If a cell has curved faces, e.g. a bubble in a soap foam, it is called the generalized polyhedron. Some generalized polyhedra have many interesting geometric properties; unfortunately, we cannot apply right-away conditions (**M2**)-(**M5**). An alternative way to characterize shape properties of a generalized polyhedron is based on the definition of a generalized pyramid.

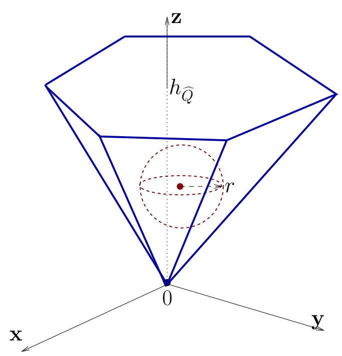

Fig. 3. A reference pyramid $\widehat{\mathsf{Q}}$ containing a sphere of radius r.

Definition 1. Let $k \geq 3$ and $\gamma_\star < 1$. A generalized pyramid Q with k lateral faces and shape-regularity constants $\gamma_\star$ and $\tau_\star$ is a subset of $\Re^3$ that can be constructed in three steps:

1. Take a pyramid $\widehat{\mathsf{Q}}$ whose base $\widehat{\mathsf{f}}$ is a convex polygon with k edges. Let $\mathsf{v}_{\widehat{\mathsf{Q}}}$ be the vertex of this pyramid, $h_{\widehat{\mathsf{Q}}}$ be its diameter, and $H_{\widehat{\mathsf{Q}}}$ be its height (see Fig. 3). Up to a rigid-body displacement, we can assume that $\mathsf{v}_{\widehat{\mathsf{Q}}}$ is in the origin and $\widehat{\mathsf{f}}$ is a subset of the plane $z = H_{\widehat{\mathsf{Q}}}$. We also assume that $\widehat{\mathsf{Q}}$ contains a sphere of radius
$$r \geq \gamma_\star\, h_{\widehat{\mathsf{Q}}}.$$
2. Define a *radial* one-to-one C^1 mapping Φ of the pyramid $\widehat{\mathsf{Q}}$ into itself. In a radial map a point $\mathbf{x}$ and its image $\mathbf{x}' = \Phi(\mathbf{x})$ lie on the same ray emanating from the origin. We assume that

$$\max_{\mathbf{x}\in\widehat{\mathsf{Q}}} \|\nabla\Phi(\mathbf{x})\| \leq \tau_\star \quad \text{and} \quad \max_{\mathbf{x}'\in\mathsf{Q}} \|\nabla(\Phi^{-1})(\mathbf{x}')\| \leq \tau_\star. \tag{4}$$

3. Define the generalized pyramid $\mathsf{Q} \equiv \Phi(\widehat{\mathsf{Q}})$. The image of the base $\widehat{\mathsf{f}}$ is a C^1 surface f, $\mathsf{f} \equiv \Phi(\widehat{\mathsf{f}})$, that we will refer to as the *base* of the generalized pyramid. Accordingly, the images of the k lateral faces of $\widehat{\mathsf{Q}}$ will be referred to as the lateral faces of Q.

The convexity assumption of $\widehat{\mathsf{f}}$ could be replaced with a star-shaped condition (**M3**).

Definition 2. A generalized polyhedron P is formed by the generalized pyramids that have the same vertex $\overline{\mathbf{x}}_{\mathsf{P}}$. The vertex $\overline{\mathbf{x}}_{\mathsf{P}}$ lies strictly inside P. The boundary $\partial\mathsf{P}$ is the union of the bases of the generalized pyramids. These bases will be referred to as the faces of P.

Now, we describe a class of shape-regular generalized polyhedral meshes. A generalized polyhedral mesh Ω_h is called shape-regular if it satisfies the following condition.

(**G1**) Every generalized polyhedron $\mathsf{P} \in \Omega_h$ is the union of at most $\mathcal{N}_\star$ generalized pyramids with at most $\mathcal{N}_\star$ lateral faces and shape constants $\gamma_\star$ and $\tau_\star$.

Condition (**G1**) is related to the mesh shape regularity conditions (**M1**)–(**M5**) introduced above. For instance, it implies immediately that every cell P is star-shaped with respect to the common vertex $\overline{\mathbf{x}}_{\mathsf{P}}$ of the generalized pyramids that form it. Boundness of the mapping Φ is critical for proving the uniform scaling (1), the Agmon inequality (2) and the approximation result (3). Finally, we can prove that the average normal vector $\widetilde{\mathbf{n}}_{\mathsf{f}}$ to a curved face f is well behaved:

$$\widetilde{\mathbf{n}}_{\mathsf{f}} = \frac{1}{|\mathsf{f}|}\int_{\mathsf{f}} \mathbf{n}_{\mathsf{f}} \mathrm{d}S, \qquad \|\widetilde{\mathbf{n}}_{\mathsf{f}}\| \geq \frac{2\gamma_\star}{\tau_\star^4}.$$

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