Tetrahedral Mesh Construction for Unit Tangent Bundle over Genus-Zero Surfaces

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Unit tangent bundle of a surface carries various information of tangent vector fields on that surface. For 2-spheres (i.e. genus-zero closed surfaces), the unit tangent bundle is a closed 3-manifold that has non-trivial topology and cannot be embedded in R^3. Therefore it cannot be constructed by existing mesh generation algorithms directly. This work aims at the first discrete construction of unit tangent bundles over 2-spheres using tetrahedral meshes. We propose a two-stage algorithm for the construction, which starts from constructing two local bundles and then combines them into a global bundle.

1 Introduction

A *unit tangent bundle* of a surface equips each point of the surface with a unit circle. It is a powerful tool in Riemannian geometry ([1]) that naturally represents all the *unit tangent vector fields* on a surface. Each unit tangent vector field on the surface can be represented as a *section* (one vector per point) of the bundle.

Despite the important role of unit tangent bundles on theoretical side, they are unfortunately missing from the literature of corresponding engineering fields (such as surface vector field design [3, 5, 2]) due to lack of an appropriate discrete representation. The goal of this work is to give the first tetrahedral mesh construction for unit tangent bundles on 2-spheres.

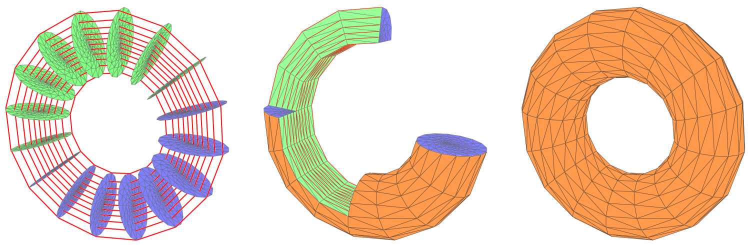
Unit tangent bundle for 2-sphere is *non-trivial* (i.e. not a direct product). Here we utilize the idea of *local trivialization.* Without loss of generality we use a standard 2-sphere centered at the origin of R^3 with radius 1 to illustrate the idea.

1. Slice the sphere open along the equator into two semi-spheres and map them to two unit disks (i.e. *covering disks*) on the XY plane by stereographic projection.

2. For each covering disk we build the local bundle, which is a direct product (i.e. trivial) of disk and circle, namely a solid torus.
3. Glue two local bundles along their boundary surface to form a global bundle, which is non-trivial. This gives us the unit tangent bundle for the input sphere. Note that the gluing here depends on the transition function between two covering disks in the first step, which is $\phi_{21}(z) = \frac{1}{z}$ in our case.

2 Local Bundle

Here we construct the unit tangent bundle T (tetrahedral mesh) for each covering disk D (triangular mesh), which is a direct product $T = D \times S^1$. We divide this task into two steps.



(a) (b) (c)

Fig. 1. A local bundle is a direct product of a disk and a circle, which is a solid torus. It can be represented by a prismatic mesh and further subdivided into a tetrahedral mesh, both with a regular structure that reflects the direct-product nature.

In the first step, we extrude disk D along circle S^1 to obtain a prismatic mesh, which is a volumetric mesh consisting of prisms. Obviously, such a mesh has a layered structure, where each layer is a direct product of disk D and a line segment.

In the second step, we subdivide the prismatic mesh into a tetrahedral mesh. The challenge here is that after the subdivision, the triangulation on the boundary surface of the volumetric mesh should have certain pattern, as required in Section 3. This boundary constrained subdivision problem has been successfully solved in [4]. That paper convert this 3D problem to an equivalent 2D graph labeling problem on the base mesh, and provided provable algorithms for all kinds of boundary conditions. Here we simply run that algorithm on each layer of our prismatic mesh, with the boundary condition specified in section 3. The output will be a tetrahedral mesh like the one shown in Figure 1(c).

3 Global Bundle

After constructing two local bundles T_1 and T_2 as in Section 2, we can combine them into a global bundle T. This is equivalent to identifying their boundary surface ∂T_1 and ∂T_2 in a special way.

We first look at the requirements that a valid global construction should satisfy.

1. According to the transition function ϕ_{21}, a vertex $v_j^2 \in \partial D_2$ with polar angle θ_j should be mapped to vertex $v_j^1 \in \partial D_1$ with polar angle $\theta_i = -\theta_j$, together with the fiber at v_j^2 (v_i^1) to the fiber at v_i^1 (v_j^2).
2. According to the Jacobian J_{21}, the circle (or fiber) at vertex $v_j^2 \in \partial D_2$ with polar angle θ_j should be rotated by $\pi - 2\theta$ after being mapped to ∂D_1.

To meet all these requirements, we should not only define an appropriate boundary triangulation of each local bundle, but also define a valid map between two boundary triangulations.

Recall the prismatic mesh we constructed for each local bundle $T_i = D_i \times S^1$ (see Section 2), its boundary surface is a quad mesh with a 2D grid structure of size $M \times M'$, where M is the number of sample points along the boundary circle ∂D_i of the base mesh D_i and M' is the number of sample points along fiber circle S^1. After the prismatic mesh is subdivided into tetrahedral mesh, the sample points on the boundary are not changed, therefore the grid structure is also preserved. Here we assign each sample point in ∂T_i with a *grid coordinate* (i, j), where $0 <= i < M$ and $0 <= j < M'$.

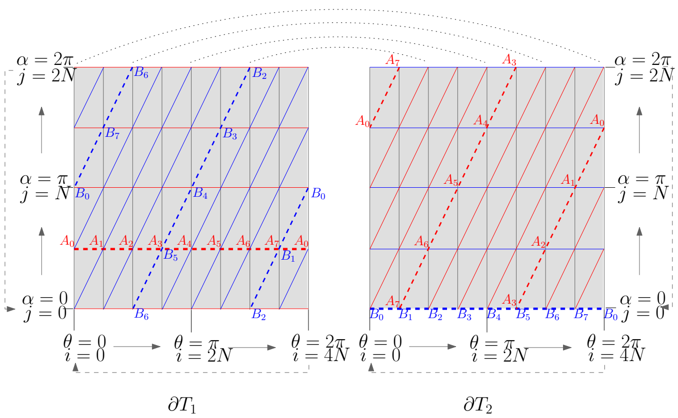


Fig. 2. The boundary triangulation and the corresponding map for global bundle construction.

With this grid structure and grid coordinates, we define the following boundary triangulation for each local bundle ∂T:

1. The vertex set forms a $4N \times 2N$ ($N \in Z^+$) grid structure;
2. Every edge connects two vertices with one of the following three types of grid coordinates: (i, j) and $(i+1, j)$, (i, j) and $(i, j+1)$, or (i, j) and $(i+1, j+1)$.

Based on the above triangulation, we define the following map $G_{21} : \partial T_2 \to \partial T_1$ to identify the boundary surfaces of two local bundles: G_{21} sends vertex (i, j) in ∂T_2 to vertex $(-i, j+N-i)$ in ∂T_1. By denoting a vertex (i, j) in mesh ∂T_k as $v^k_{(i,j)}$, the gluing map can be written as:

$$G_{21}\left(v^2_{(i,j)}\right) = v^1_{(-i,j-i+N)} \tag{1}$$

Here we claim that such a gluing map induces a bijection between the boundary surface of two local bundles. We also claim that the resulting tetrahedral mesh satisfies all the requirements for a discrete representation of unit tangent bundle over 2-sphere.

4 Conclusion and Remarks

This paper propose the first discrete representation for unit tangent bundles over 2-spheres using tetrahedral meshes. This is carried out by building local bundles over covering disks and then combining them into a global bundle for the whole sphere. The construction is guaranteed to generate the desired results.

Note that our boundary triangulation in global construction requires that the partitioning cycle (which is piecewise linear) on the original surface should consist of $4N$ edges for some positive integer N. If the initially picked cycle does not satisfies this, one can always perturb it or subdivide some edges on it to make it qualify.

Another note is that, for local bundles, the tetrahedral meshes T_i has an embedding in R^3, and also has a Euclidean metric induced from the embedding. For the global bundle, our construction only guarantees that it is topologically faithful, without assigning any meaningful embedding or metric yet. However, it is possible to assign the latter to the tetrahedral mesh we construct, which will be an interesting direction to explore in the future.

References

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