Solving Stokes equation in plane irregular regions using an optimal consistent finite difference scheme

J. G. Tinoco-Ruiz, F. Domínguez-Mota*, S. Mendoza-Armenta and G. Tinoco-Guerrero

Facultad de Ciencias Físico Matemáticas
Universidad Michoacana de San Nicolás de Hidalgo
Edificio "B", Ciudad Universitaria, Morelia, México 58040, *dmota@umich.mx

Summary. Historically, finite difference Schemes (FDS) defined in logically rectangular grids have been widely used to get numerical approximations to the solution of partial differential equations in simple domains, *i.e.*, rectangular regions or those suitable to be decomposed in rectangles, but when the region is not of this kind, the classical schemes can not longer be applied. However, the development of efficient methods for meshing irregular planar regions using quadrilateral elements allows new schemes to be defined. In this paper, in order to solve numerically Stokes equation on an irregular domain using finite differences, we show the application of a simple scheme derived from a local optimization problem.

Key words: Stokes differential equation, finite difference scheme, numerical grid generation

1 Introduction

Finite difference schemes on rectangular regions follow with ease from Taylor's theorem. However, its application to irregular domains requires a suitable structured convex grid. Trying to overcome this problem, irregular regions have been often approximated by block-rectangular regions, but for many actual domains this is a rather poor representation. An approach which preserves the shape of the domain is given by coordinate transformations between the physical region and a rectangle, which yields a transformed equation on the latter whose solution can be approximated using the classical finite difference schemes. Unfortunately, explicit changes of coordinates are only known for simple, academic regions, and this is a serious limitation to the method.
More satisfactory results can be obtained for very irregular regions using differences defined through structured grids generated by the variational grid

generation method. This method consists of minimizing an appropriate functional [1]. Area and harmonic functionals can be used for gridding a wide variety of simple connected domains in the plane [2], whose boundaries are closed polygonal Jordan curves with positive orientation.
Using these kind of structured convex grids, some authors have designed schemes for the discretization of partial derivatives directly on the physical region; we can mention for instance the works due to Steinberg, Shashkov, Hyman and Castillo [3], and Tinoco et al [4]. Extending this idea, in [5] we designed a new direct finite difference scheme for the numerical solution of Poisson's equation on irregular regions whose performance turned out to be quite satisfactory. In this work, we apply this scheme in order to calculate the numerical solution of Stokes equation.

2 Discretization of Stokes equation

Let us consider first an elliptic boundary value problem defined on a bounded, simply connected, plane region Ω

$$\begin{aligned} Lu = Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y = F \\ u|_{\partial\Omega} = u_0, \end{aligned} \tag{1}$$

where the functions A, B, C, D, E, F depend on the variables x, y, and u_0 is the value of the unknown function at the boundary of the region. A nine-point finite difference scheme for (1) at the point $p \in \Omega$ is a a linear combination of the values of u at the points $\{q_{0,p}, q_{1,p}, \ldots, q_{k,p}\}$ [6]

$$L_0(p) = \Gamma_{0,p} u(q_{0,p}) + \Gamma_{1,p} u(q_{1,p}) + \cdots + \Gamma_{8,p} u(q_{8,p}), \tag{2}$$

such that the difference $\delta_p = L_0(p) - [Lu]_p$ is small.
Expanding $[Lu]_p$ at p, the proposed finite difference scheme is obtained by solving the least square problem

$$\begin{aligned} &\min R_6^2 + R_7^2 + R_8^2 + R_9^2 \\ &\text{s. t. } R_i = 0, \quad i = 0, \cdots, 5. \end{aligned} \tag{3}$$

where

$$R_0 = \sum_{k=0}^{8} \Gamma_{k,p}, \quad R_1 = \sum_{k=0}^{8} \Gamma_{k,p} \Delta x_{k,p} - D, \quad R_2 = \sum_{k=0}^{8} \Gamma_{k,p} \Delta y_{k,p} - E,$$

$$R_3 = \sum_{k=0}^{8} \Gamma_{k,p} \Delta x_{k,p}^2 - 2A, \; R_4 = \sum_{k=0}^{8} \Gamma_{k,p} \Delta x_{k,p} \Delta y_{k,p} - B,$$

$$R_5 = \sum_{k=0}^{8} \Gamma_{k,p} \Delta y_{k,p}^2 - 2C,$$

are the order one and two residuals, Δx_i, Δy_i are the x and y components of $q_{k,p} - p$, and

$$R_6 = \frac{1}{6}\sum_{k=0}^{8} \Gamma_{k,p}\Delta x_{k,p}^3 \qquad R_7 = \frac{1}{2}\sum_{k=0}^{8} \Gamma_{k,p}\Delta x_{k,p}^2 \Delta y_{k,p}$$

$$R_8 = \frac{1}{2}\sum_{k=0}^{8} \Gamma_{k,p}\Delta x_{k,p} \Delta y_{k,p}^2 \qquad R_9 = \frac{1}{6}\sum_{k=0}^{8} \Gamma_{k,p}\Delta y_{k,p}^3$$

are the third order residuals.

Once the difference scheme defined by equation (3) is available, it is possible to generate schemes for first and second order differential equations. The numerical solution of Poisson's equation was addressed in [5], and its application is very simple.

Stokes equations are given by

$$-\nabla^2 \mathbf{U} = \nabla P + \mathbf{F}, \tag{4}$$

$$\nabla \cdot \mathbf{U} = 0, \tag{5}$$

where u and v are the velocity components of the fluid, $\mathbf{U} = (u, v)^T$, P is the pressure, and f_1 and f_2 are the components of the applied body force, and $\mathbf{F} = (f_1, f_2)^T$. It describes the type of fluid flow where advective inertial forces are small when compared with viscous forces, a situation in flows where the fluid velocities are very slow, the viscosities are very large, or the length-scales of the flow are very small.

At every point $p_{i,j}$ of the set of inner grid points $\{p_{i,j}\}$, there are three unknowns, $u_{i,j}$, $v_{i,j}$ and $P_{i,j}$. However, even though it is possible to discretize (4) and (5) directly, due to inf-sup condition [7], it is not convenient to approximate the velocity field and the pressure simultaneously using the second order finite differences defined by (3), since they are closely related to bilinear finite elements on triangular elements. A better strategy is to rewrite Stokes equation first.

Divergence of (4) and condition (5) yield $\nabla^2 P = 0$. This is a Laplace problem whose solution can be approximated by means of

$$\mathbf{D}\hat{P} = \mathcal{F}_{\mathcal{P}} \tag{6}$$

where[1] $\hat{P} = (P_{1,1}, \cdots, P_{mn})^T$, $\mathbf{D}$ is the matrix representation of $-\nabla^2$, obtained by solving the optimization problem (3) with $A = -1$, $B = 0$, $C = -1$, $D = 0$, and $E = 0$ at every point of $\mathcal{P}$, and $\mathcal{F}_{\mathcal{P}}$ collects the evaluation of the forces f_1 and f_2 as well as the boundary information.

Once the approximation to P is known at every grid point, ∇P can be approximated by choosing A, B, C, D and E in order to produce $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$. These problems are Poisson-like, and the same difference scheme can be applied to obtain the approximation to u and v.

[1]This is vector of all the elements of $\{p_{i,j} : 1 < i < m, 1 < j < n\}$, regarded as a single column.

3 Numerical Tests

For the numerical tests, we selected 2 polygonal regions, both of them approximations to real geographical locations. They will be denoted as Havana bay (hab), and Michoacán (mic) (See figure 1). They were scaled and shifted to lie in $[0,1] \times [0,1]$. For these boundaries, structured grids with 21, 41 and 81 points were generated with UNAMALLA [8] by minimizing the functional $H_\omega - A$ with default parameters [2].
The closed form solution of Stokes equation was defined to be the zero force Papkovich-Neuber solution [9] given by

$$\begin{aligned} u &= -1/2\,x^3 + 9/2\,x\,y^2 + 1/2\,x\,\left(3\,x^2 - 3\,y^2\right) + 1/2\,e^x \sin y, \\ v &= -9/2\,x^2 y + 1/2\,y^3 + 1/2\,y\,\left(3\,x^2 - 3\,y^2\right) + 1/2\,e^x \cos y, \\ P &= 6\,x^2 - 6\,y^2. \end{aligned}$$

These were also the values for the Dirichlet condition.
The sparse systems obtained in each case were solved by sparse Gaussian Elimination; the quadratic error norms $\|\cdot\|_2$ for the tests were calculated as the grid functions

$$\|\mathcal{U} - \hat{\mathcal{U}}\|_2 = \sqrt{\sum_{i,j} (\mathcal{U}_{i,j} - \hat{\mathcal{U}}_{i,j})^2 \mathcal{A}_{i,j}}, \tag{7}$$

where $\mathcal{U} = (\mathcal{U}_{i,j})$ and $\hat{\mathcal{U}} = (\hat{\mathcal{U}}_{i,j})$ are the exact and approximated solution at the i,j^{th}-grid node respectively, and $\mathcal{A}_{i,j}$ is the area of the i,j-element. The empirical orders O_u, O_v and O_P between two consecutive grid orders were calculated according to the formula

$$\log\left(E_i/E_j\right) / \log\left(n_j/n_i\right), \tag{8}$$

where E_i is the quadratic error associated to the numerical solution calculated with a grid with n_i points per side. Errors and orders are summarized in table 1. One must note that the irregularity of the selected boundaries is reflected in a slight loss of order, but the same fact is the main argument to conclude that the approximations obtained are satisfactory; the numerical results presented show that, the natural extension of the discretization addressed in [5] for Poisson's equation, can indeed be used with ease to approximate the solution of Stokes equation on very irregular regions by means of a simple scheme.

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Michoacán Havana bay

Fig. 1. Test regions.

Table 1. Quadratic error for the test problem

Region	$\|u - \hat{u}\|_2$	O_u	$\|v - \hat{v}\|_2$	O_v	$\|P - \hat{P}\|_2$	O_P
hab21	1.0175E-03		1.1855E-03		1.0552E-03	
hab41	3.0171E-04	1.82	4.0434E-04	1.61	3.8954E-04	1.49
hab81	1.1312E-04	1.44	1.2845E-04	1.68	1.0702E-04	1.90
mic21	1.0561E-03		8.9833E-04		1.6506E-03	
mic41	2.7614E-04	2.01	2.6896E-04	1.80	6.5440E-04	1.38
mic81	1.2924E-04	1.12	8.1268E-05	1.76	3.1795E-04	1.06

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