Label-Invariant Mesh Quality Metrics

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Abstract. Mappings from a master element to the physical mesh element, in conjunction with local metrics such as those appearing in the Target-matrix paradigm, are used to measure quality at points within an element. The approach is applied to both linear and quadratic triangular elements; this enables, for example, one to measure quality within a quadratic finite element. Quality within an element may also be measured on a set of *symmetry points*, leading to so-called *symmetry metrics*. An important issue having to do with the labeling of the element vertices is relevant to mesh quality tools such as Verdict and Mesquite. Certain quality measures like area, volume, and shape should be *label-invariant*, while others such as aspect ratio and orientation should not. It is shown that local metrics whose Jacobian matrix is non-constant are label-invariant only at the center of the element, while symmetry metrics can be label-invariant anywhere within the element, provided the reference element is properly restricted.

1 Measuring Quality *Within* Mesh Elements

Mesh quality is important for maintaining accuracy and efficiency of numerical simulations based on the solution of partial differential equations [6]. Mesh quality metrics are used to measure mesh quality and there is an extensive literature on the subject, particularly for finite element meshes [8], [14], [16], [17], [19]. For simplicial elements, 'shape' is an important measure [3], [15]. A shape measure based on condition number was proposed in [4], [5]. In [2] and [7] the notion of shape for simplicial elements was formalized. Few works discuss quality measures for quadratic elements [1], [18]; the latter reference being the only example that goes beyond detecting singular points. Significantly, the latter is limited to triangle elements.

Engineers usually measure mesh quality by one of two basic approaches, depending on whether they are working with unstructured or structured meshes. The quality of an unstructured mesh is most often studied in terms of the individual elements within the mesh. Elements are most often polygons or polyhedra, with triangles, tetrahedra, quadrilaterals, hexahedra, prisms, and pyramids being the most commonly used types. A mesh element contains vertices and/or nodes, usually given in some canonical ordering. The vertices/nodes have coordinates $x_m \in R^d$, with $d = 2, 3$ and $m = 0, 1, 2, ..., M$, with M depending on the element type and order. The quality q_ε of an element is most often defined as some continuous function of the element coordinates.

Triangular element *aspect ratio*, given by the formula $q_\varepsilon = \frac{L_{max}}{2\sqrt{3}\,r}$ is an example of the first approach to measuring quality. Because the lengths in the formula depend on the coordinates of the vertices in the triangle, the element metric is a function of the vertex coordinates. The formula only applies to straight-sided (low-order) triangles.

The second approach to measuring mesh quality arises in the structured meshing community. A global mapping from a logical block U to a physical block $\Omega \subset R^d$ is found and serves to define a discrete grid. When $d = 3$, the map takes the form $x = x(\Xi)$, with $\Xi = (\xi_1, \xi_2, \xi_3) \in U$ and $x = (x_1, x_2, x_3) \in \Omega$. The tangents to the map, $dx_i/d\xi_j$, $i, j = 1, 2, 3$, are used to define local mesh quality at a point within the domain. For example, for $d = 2$, one measures *orthogonality* at a point in U via the *local* metric $x_{\xi_1} \cdot x_{\xi_2}$.

Over the past decade, the author has used a third approach to measuring quality that is a hybrid of the two basic approaches [8], [9]. For each *element* of a mesh, let there be a map from a logical (or master) element to the physical element. Then one can measure local quality *within* the element using formulas based on the local tangents of the map, just as is done in the structured meshing community. Because the element map depends on the coordinates of the vertices/nodes within the element, the local quality at a point within the element also depends on these coordinates. Although the third approach uses the master element concept from the finite element method, it can be used to measure quality whether or not the mesh is intended to be used in a finite element simulation. That is, measuring quality by the third approach applies equally well to finite element, finite volume, finite difference, or even spectral element simulations, as is the case with the first approach.[1]

The third approach does not preclude the measurement of *element* quality, if desired. Let μ be a local quality metric and $\mu(\Xi_n)$, $n = 1, ..., N$, be the local qualities measured at N points Ξ_n within the master element. Then element quality may be defined to be, for example, $q_\varepsilon = \max_n\{\mu(\Xi_n)\}$, $q_\varepsilon = \min_n\{\mu(\Xi_n)\}$, or the p^{th} power-mean, $p \neq 0$, of the local qualities:

$$q_\varepsilon = \left(\frac{1}{N} \sum_{n=1}^{N} [\mu(\Xi_n)]^p \right)^{1/p} \tag{1}$$

with $\mu > 0$. The power-mean, minimum, and maximum are attractive as a means to combine the local metrics because the range of the resulting element metric is the same as the range of the local metric.

[1] For the sake of clarity, we propose to call the first approach to measuring quality the 'element' quality method, the second approach the 'pointwise' quality method, and the third approach the 'hybrid' quality method.

Three examples are given to show why this third approach may be attractive. First, consider a planar quadrilateral element, with *area* as the quantity of interest. Let the four vertices be labeled x_m, $m = 0, 1, 2, 3$, in counter-clockwise order. The linear map is $x(\xi_1, \xi_2)$ and the Jacobian matrix $A(\xi_1, \xi_2)$

$$A = [\,(x_1 - x_0) + (x_0 - x_1 + x_2 - x_3)\,\xi_2,\ (x_3 - x_0) + (x_0 - x_1 + x_2 - x_3)\,\xi_1\,]$$

The signed area at any given point within the element is $\alpha = det(A)$.[2] In the 'element' quality method, a standard quadrilateral area measure is

$$A_1 = \frac{1}{2}\,det([x_1 - x_0, x_3 - x_0]) + \frac{1}{2}\,det([x_3 - x_2, x_1 - x_2]) \tag{2}$$

In the 'pointwise' quality method, an area measure based directly on the local metric $\alpha(\xi_1, \xi_2)$ is

$$A_2 = \min\{\,\alpha(0,0), \alpha(1,0), \alpha(1,1), \alpha(0,1)\,\} \tag{3}$$

To compare these two area measures, consider the quadrilateral with vertex coordinates $x_0 = (0, 0)$, $x_1 = (1, 0)$, $x_2 = (\frac{1}{4}, \frac{1}{4})$, and $x_3 = (0, 1)$[3] Then formula (2) yields $A_1 = \frac{1}{4}$, while formula (3) gives $A_2 = -\frac{1}{2}$. Therefore, the latter formula detects the negative Jacobian, while the former does not.[4]

In the second example, consider the quality of a high-order finite element such as a quadratic triangle having three mid-side nodes. With the exception of [18], there are no examples in the literature that measure the quality of a quadratic finite element by the 'element' quality method and this reference is limited to triangular elements. The quality of high-order finite elements such as tetrahedra and hexahedra can be assessed using the 'hybrid' quality method. In fact, the method is exactly the same as it is for linear elements: evaluate the local quality metric at a point by computing the Jacobian of the relevant map from the master to the physical element and combine the local qualities via the formulas for maximum or minimum quality or the power-mean (1). *Although this method often does not bound the worst quality within the element, a judicious choice of sample points within the element can provide a lot of useful information.* For a quadratic triangle, for example, it is reasonable to measure the local quality at the three corner vertices and at the three mid-side nodes. In this manner one can measure local shape, size, and orientation within an element using, for example, metrics from the Target-matrix paradigm [10], [11], [12].

In the third example, suppose one wanted to optimize the quality of a locally-refined mesh containing, as a submesh, the two linear quadrilaterals

[2] The notation $A = [x_{\xi_1}, x_{\xi_2}]$ signifies that the first column of A is the $1 \times d$ vector x_{ξ_1} and that the second column of A is the $1 \times d$ vector x_{ξ_2}. Similar notation is used throughout. Also, $det(A)$ signifies the determinant of A.

[3] This poor quality quadrilateral is called an *arrow* due to the re-entrant corner.

[4] Formula (2) can also be written in terms of local metrics. In this example, the ability to detect negative Jacobians is a matter of choosing the minimum instead of the linear average.

on the right in Figure 1 and one quadratic quadrilateral on the left. An objective function could be based on (1), for example, in which μ is any desired local quality metric. The points Ξ_n would include the logical corners of the three quadrilaterals, along with the logical points corresponding to the mid-edge nodes in the quadratic 'quadrilateral'. If the mid-side node is allowed to be 'free' in the optimization, then the quadratic map is required for the left quadrilateral, while if the mid-side node is constrained to the mid-point of the straight edge, then only a linear map is needed. Note that in the hybrid quality method one can have more than one quality measurement per vertex within the mesh.

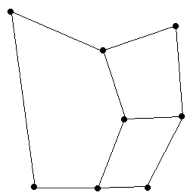

Fig. 1. Three non-conformal quadrilateral elements

The hybrid method is clearly more flexible than the element quality method. It becomes yet more powerful when combined with concepts from the Target-matrix paradigm which provides numerous referenced local metrics. Figure 2 shows the basic idea: let there be maps from the logical element to the physical element, and from the logical element to a reference element which gives the desired element configuration. Let the Jacobian matrix of the first map be denoted by $A(\Xi)$ and the Jacobian matrix of the second map be $W(\Xi)$. It is reasonable to assume that the reference element is non-degenerate; in that case, W is non-singular. To compare the two matrices, form $T = AW^{-1}$ so that when $A = W$, $T = I$. The quality at a point Ξ within the element is given by a quality metric $\tilde{\mu}(\Xi) = \mu[T(\Xi)]$. A variety of useful local quality metrics $\mu(T)$ are studied in [11]. Most of the quality metrics are combinations of the fundamental quantities $\tau = det(T)$, $|T|$, $|T^tT|$, $tr(T)$, and $|T^{-1}|$, so the analysis to follow is focused on these.

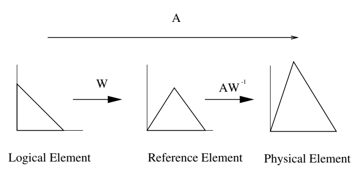

Fig. 2. Relation between the Logical, Reference, and Physical Elements

2 Label-Invariance of Quality Metrics

An important practical issue arises in the measurement of a priori unstructured mesh quality that has to do with the labeling of the vertices within an element. Consider the logical (left) and physical (right) triangles in Figure 3. The vertices in the logical triangle are labeled 0,1,2 in counter-clockwise order, while the vertices in the physical triangle are labeled m, $m+1$, $m+2$, again in counter-clockwise order. If $m = 0$, then physical vertex m corresponds to logical vertex 0, physical vertex $m+1$ to logical vertex 1, and so on. However, if $m = 1$, then physical vertex m corresponds to logical vertex 1, physical vertex $m+1$ to logical vertex 2, and so on. In most unstructured mesh generation software, the value of m is determined by the order of the nodes in the list of nodes for the given element. As an example, in the Verdict mesh quality assessment code [19], one step in calculating the quality of an element is to obtain the list of vertices that are contained by the element. No sorting of this list is done and so the first vertex in the list automatically becomes the image vertex 0, and the second vertex in the list becomes image vertex 1, etc. The impact of this can be seen in the two examples to follow.

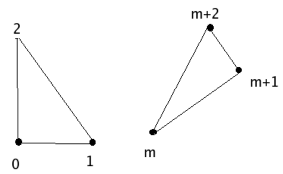

Fig. 3. Vertex Labeling Choices

First, consider the metric A_1 in (2). If the indices are cyclically permuted by 1 the formula becomes

$$A_1' = \frac{1}{2}\,det(x_2 - x_1, x_0 - x_1) + \frac{1}{2}\,det(x_0 - x_3, x_2 - x_3) \tag{4}$$

One can show that $A_1' = A_1$, that is, the element area is independent of the choice of labeling of the vertices. This is an example of what we will call a label-invariant metric. Not all element metrics enjoy this property. For example, let

$$L_h = \tfrac{1}{2}\left\{|x_1 - x_0| + |x_2 - x_3|\right\}, \; L_v = \tfrac{1}{2}\left\{|x_3 - x_0| + |x_2 - x_1|\right\}$$

and define the quadrilateral aspect ratio metric to be $AR = L_v/L_h$. Then the cyclically permuted formula is $(AR)' = 1/(AR)$.

Definition 1. Label-invariance for Element Quality Metrics
An element quality metric is *label-invariant* if, for an arbitrary physical element, its value is the same no matter which corner vertex of the physical element is labeled zero.

The question arises as to whether or not quality metrics should be label-invariant. In general, the answer is no because while metrics such as area, volume, and shape should probably be label-invariant, metrics like aspect ratio may be more informative if they are not label-invariant.

The labeling issue above was described within the context of the first approach to the measurement of element quality. It also exists within the context of the third approach, with a few additional twists, due to the fact that the mapping from the logical to the physical element depends on the choice of labeling and the Jacobian matrix thus depends on m. The first twist in the third approach is that even the local metric at a point may or may not be label-invariant, so that one can speak of label-invariant local metrics in addition to label-invariant element metrics. Second, the label-invariance may depend on the choice of reference element. For example, if an isotropic reference element is selected, it is more likely that the local metric is label-invariant. Third, label-invariance of a local metric may depend on the point within the element at which it is evaluated. This necessitates a modification of the previous definition for the case of measuring quality within elements.

Definition 2. Label-invariance for Local Quality Metrics
Let $\mu_m(\Xi) = \mu(\,T_m(\Xi)\,)$ be a local (target-matrix) quality metric. Let the reference element be a particular type and have a particular configuration within that type. Then the local quality metric is label-invariant at Ξ (with particular reference) if, for an arbitrary physical element (whose type is the same as the reference element), $\mu_m(\Xi)$ is a constant for all m.

In additional to the above definition, there is another concept of importance that arises in the third approach to measuring quality. Let $\{\Xi^{(0)}, \ldots, \Xi^{(N)}\}$ be a collection of *symmetry points* within the master element.[5] The symmetry points are each functions of Ξ. Define a non-local *symmetry metric* $\sigma_m(\Xi)$, similar to (1), based on an associated local metric μ. For example, in terms of the power-mean

$$\sigma_m(\Xi) = \left(\frac{1}{N} \sum_{n=0}^{N} [\,\mu_m(\Xi^{(n)})\,]^p \right)^{1/p} \tag{5}$$

and for the minimum and maximum

$$\sigma_m(\Xi) = \min_n \{\mu_m(\Xi^{(n)})\} \tag{6}$$

$$\sigma_m(\Xi) = \max_n \{\mu_m(\Xi^{(n)})\} \tag{7}$$

Definition 3. Label-invariance for Symmetry Quality Metrics
Let σ_m be a symmetry metric derived from the metric μ_m. Let the reference element be a particular type and have a particular configuration within that

[5] It will become apparent later what is meant by a symmetry point.

type. Then the symmetry metric is label-invariant at Ξ (with particular reference) if, for any arbitrary physical element (whose type is the same as the reference element), σ_m is a constant for all m.

It is noted that the concept of label-invariance is not the same as the concept of orientation-invariance. As a example, the aspect ratio metric AR is orientation-invariant because, if the element is rigidly rotated, the value of the metric does not change; in contrast, the metric is not label-invariant.

The comments and definitions presented in this section should become clearer in the sections to follow, where the hybrid quality method is studied on triangles with linear and quadratic maps.

3 Linear Planar Triangles

3.1 The Linear Map

Let $\Xi = (\xi, \eta)$ and $U = \{\Xi \,|\, \xi \geq 0\,, \eta \geq 0\,, \xi + \eta \leq 1\}$ be a logical triangle. Let x_0, x_1 and x_2 be the three (ordered) vertices of a physical triangle in R^2. For linear triangles in the xy-plane, the mapping from U to the physical triangle is

$$x(\Xi) = x_0 + (x_1 - x_0)\xi + (x_2 - x_0)\eta \tag{8}$$

Then, $x_\xi = x_1 - x_0$ and $x_\eta = x_2 - x_0$, so the Jacobian matrix is $A = [x_\xi, x_\eta]$. For the linear triangle map, the Jacobian matrix and its determinant, $det(A)$, are independent of ξ and η and are thus constant over the element (i.e., the same at every point in U).

3.2 The Reference Element

Let a reference triangle with vertex coordinates w_0, w_1, and w_2 be given. The Jacobian matrix W of the reference triangle is obtained from the previous relations by replacing x_0 with w_0, x_1 with w_1, and x_2 with w_2, yielding $W = [w_1 - w_0, w_2 - w_0]$. The reference element is assumed to be non-degenerate, i.e., $det(W) \neq 0$; therefore W^{-1} exists. Let $T = AW^{-1}$ be the weighted Jacobian matrix. Both W and T are constant over the linear triangle.

Let $\rho > 0$, R be any 2×2 rotation matrix, and

$$V = \begin{pmatrix} 1 & 1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix} \tag{9}$$

Then if the reference triangle is equilateral, the matrix W representing the reference Jacobian belongs to the set of 2×2 matrices $\mathcal{M}$ of the form $\rho\, R\, V$.

3.3 Label-Invariance

The map (8) assigns the vertex $(0, 0) \in U$ to the image vertex 0 in the physical triangle. In general, the vertex $(0, 0)$ could have been assigned to any of the image vertices 0, 1, or 2 (see Figure 3). To preserve orientation, it is assumed that if reference vertex $(0,0)$ is assigned to image vertex m, with $m = 0, 1, 2$, then reference vertex $(1,0)$ is automatically assigned to image vertex $m+1$, and $(0,1)$ to image vertex $m+2$. There are thus three ways one can define a properly oriented mapping for a linear triangle, depending on which physical vertex, m is selected to be the image of $\Xi = 0$. The previous map (8) is modified to emphasize this dependence. Define the map

$$x(\Xi, m) = x_m + (x_{m+1} - x_m)\,\xi + (x_{m+2} - x_m)\,\eta \tag{10}$$

where $m = 0, 1,$ or 2.[6] Quantities derived from the map, such as the Jacobian matrices and quality metrics, will also depend on m in general.

Let the Jacobian matrices of the map (10) be A_m. From the map it is clear that $A_m = [x_{m+1} - x_m,\, x_{m+2} - x_m]$. The three Jacobian matrices are, in general, not equal to one another, and thus A_m is not a label-invariant quantity. It is straightforward to show that the Jacobian matrices obey the relation

$$A_{m+1} = A_m\, P \tag{11}$$

where P is the constant matrix

$$P = \begin{pmatrix} -1 & -1 \\ +1 & 0 \end{pmatrix} \tag{12}$$

Accordingly, $A_1 = A_0\, P$ and $A_2 = A_1\, P = A_0\, P^2$. Because $det(P) = 1$, $det(A_0) = det(A_1) = det(A_2)$. Thus $det(A_m)$ is a label-invariant quantity.

For this map, there are three weighted Jacobian matrices $T_m = A_m W^{-1}$, for which $det(T_0) = det(T_1) = det(T_2)$. Thus the local metric $\tau_m = det(T_m)$ for the linear planar triangle is label-invariant *for any choice of* Ξ *or* W.

Proposition 1. The local quantities $|T_m|$ and $|(T_m)^t(T_m)|$ are label-invariant for arbitrary Ξ if and only if $W \in \mathcal{M}$. The quantity $tr(T_m)$ is not label-invariant for any choice of W.

Proof. Suppose that $W \in \mathcal{M}$. Then let $W = \rho\, R\, V$. One can show by direct calculation that the matrix $V\, P^m\, V^{-1}$ is a rotation. Therefore, $W\, P^m\, W^{-1} = R\,(V\, P^m\, V^{-1})\, R^t$ is also a rotation. Let $Q_m = W\, P^m\, W^{-1}$. Then, $T_m = A_m\, W^{-1} = A_0\, P^m\, W^{-1} = A_0\, W^{-1}\, Q_m$. Because the Frobenius norm is invariant to orthogonal matrices,

$$|T_m| = |A_0 W^{-1} Q_m| = |A_0 W^{-1}| = |T_0|$$

The steps in this proof are reversible, so if $|T_m|$ is label-invariant, then $W \in \mathcal{M}$. The proof for $|(T_m)^t(T_m)|$ is similar. Since the trace is not invariant to a rotation, it is not label-invariant. §

[6] All vertex subscripts in this section are modulo 3 so, for example, $x_4 = x_1$.

Corollary. Any local metric $\mu_m = \mu(T_m)$, which is a function of the quantity τ_m only (e.g. $\mu(T) = \tau^2$), is label-invariant for any Ξ and W. Any local metric which is a combination of only τ_m, $|T_m|$, and/or $|(T_m)^t(T_m)|$ is label-invariant provided $W \in \mathcal{M}$. For example, the local inverse mean ratio metric $\mu(T) = \frac{|T|^2}{2\,\tau}$ (which when $d = 2$ is the same as the condition number metric) is label-invariant when $W \in \mathcal{M}$. Invariance is desirable in this case since mean ratio is intended to measure the local shape within a triangle relative to the reference shape.[7] Any local metric containing $tr(T_m)$ is not label-invariant. For example, the metric $\mu(T) = |T - I|^2$, which is the same as $|T|^2 - 2\,tr(T) + 2$, is not label-invariant. Lack of invariance is acceptable for this metric since it is intended to control the orientation within mesh elements [11]. §

These results, of course, apply to the linear map (10) and must be re-examined when the map is different.

4 Quadratic Planar Triangles

4.1 The Quadratic Map

There are three ways one can define the mapping for a quadratic planar triangle, depending on which vertex, m ($m = 0, 1, 2$) is selected to be the image of $\Xi = 0$. Write the quadratic map on U as

$$\begin{aligned} x(\Xi, m) = {} & c_{0,m} + c_{1,m}\,\xi + c_{2,m}\,\eta \\ & + c_{3,m}\,\xi^2 + c_{4,m}\,\xi\eta + c_{5,m}\,\eta^2 \end{aligned} \tag{13}$$

The tangent vectors of the map are[8]

$$x_\xi(\Xi, m) = c_{1,m} + 2\,c_{3,m}\,\xi + c_{4,m}\,\eta \tag{14}$$

$$x_\eta(\Xi, m) = c_{2,m} + c_{4,m}\,\xi + 2\,c_{5,m}\,\eta \tag{15}$$

In contrast to the linear map, the tangent vectors for the quadratic map depend on Ξ, thus the Jacobian matrix $A_m = A_m(\Xi)$ depends on Ξ. It is easy to show that the Jacobian matrix A_m for the quadratic map is constant (i.e., independent of Ξ) if and only if the physical triangle has straight sides.

Recall the relations between the logical, reference, and physical elements shown in Figure (2). When the map from the logical to the physical element is quadratic, there is no reason why the map from the logical to the reference element cannot also be quadratic. In that case, $W = W(\Xi)$, i.e, the reference Jacobian matrix can vary from one position to another. The discussion that follows does not assume W is constant so that reference elements having

[7] Because the Jacobian of the linear triangle map is constant, mean ratio also measures the shape of the triangle itself.

[8] To save space, the formulas for the coefficients in terms of the element nodes is not given since they are well-known.

curved sides are allowed.[9] Define $T_m(\Xi) = A_m(\Xi)\,[W(\Xi)]^{-1}$ and $\tau_m = det(T_m)$.

4.2 Symmetry Points for Maps to Triangular Elements

Recall that for both the linear and quadratic triangular elements there is a map x of the form $x(\Xi, m)$, where $m = 0, 1, 2$ denotes the vertex of the triangle that serves as the base in the construction the map. Let $X^{(0)}$ be an arbitrary point in the physical triangular element. Setting $X^{(0)} = x(\Xi^{(0)}, m)$ for some fixed choice of m, we have $\Xi^{(0)}$ as the pre-image of $X^{(0)}$. For each such point $X^{(0)}$ in the triangle, there are two additional points $X^{(1)} = x(\Xi^{(1)}, m)$ and $X^{(2)} = x(\Xi^{(2)}, m)$ with pre-images $\Xi^{(1)}$ and $\Xi^{(2)}$ which we define by the relations[10]

$$x(\Xi^{(1)}, m) = x(\Xi^{(0)}, m + 1) \tag{16}$$

$$x(\Xi^{(2)}, m) = x(\Xi^{(0)}, m + 2) \tag{17}$$

The points $X^{(k)}$ and their pre-images $\Xi^{(k)}$ $(k = 0, 1, 2)$ are points of symmetry because point $X^{(k+1)}$ can be obtained either from the map based at vertex m evaluated at $\Xi^{(k+1)}$ or from the map based at vertex $m + 1$ evaluated at $\Xi^{(k)}$. The symmetry points are defined by the relations above and hold on any triangle of any shape and includes both linear and quadratic maps.[11]

The relations above can be used to find the pre-image points $\Xi^{(k)}$ in terms of $\Xi^{(0)}$. Solving for $\Xi^{(1)} = (\xi^{(1)}, \eta^{(1)})$ and $\Xi^{(2)} = (\xi^{(2)}, \eta^{(2)})$, one obtains

$$\Xi^{(1)} = (1, 0) + P\,\Xi^{(0)} \tag{18}$$

$$\Xi^{(2)} = (0, 1) + P^2\,\Xi^{(0)} \tag{19}$$

where P is the matrix given in (12). Explicitly, $\Xi^{(0)} = (\xi^{(0)}, \eta^{(0)})$, $\Xi^{(1)} = (1 - \xi^{(0)} - \eta^{(0)}, \xi^{(0)})$, and $\Xi^{(2)} = (\eta^{(0)}, 1 - \xi^{(0)} - \eta^{(0)})$. Notably, the logical symmetry points do not depend on the vertices x_m, x_{m+1}, and x_{m+2} of the triangle.

4.3 Symmetry Relation for Jacobian of the Quadratic Map

Proposition 2. Let the Jacobian of the quadratic map (13) at the point Ξ be given by $A_m(\Xi) = [x_\xi(\Xi, m), x_\eta(\Xi, m)]$, with the latter given in (14)-(15). Then, for $k, m = 0, 1, 2$, the following relations hold

$$A_m(\Xi^{(k)}) = A_0(\Xi^{(k+m)})P^m \tag{20}$$

with P defined as in (12).

[9] Note, however, that W does not depend on m since there is no labeling issue with the reference element.

[10] More generally, one can write $x(\Xi^{(r+s)}, m) = x(\Xi(s), m + r)$ with $s = 0, 1, 2$, which leads to the same symmetry points.

[11] The indices k are cyclic with period 3.

Proof. The logical symmetry points derived in the previous section can be regarded as functions of $\xi^{(0)}$ and $\eta^{(0)}$. Differentiation of the pre-image formulas with respect to these variables, one finds

$$\begin{pmatrix} \frac{\partial \xi^{(k)}}{\partial \xi^{(0)}} & \frac{\partial \xi^{(k)}}{\partial \eta^{(0)}} \\ \frac{\partial \eta^{(k)}}{\partial \xi^{(0)}} & \frac{\partial \eta^{(k)}}{\partial \eta^{(0)}} \end{pmatrix} = P^k \tag{21}$$

From the relations (16)-(17) that define the symmetry points of the map, one can deduce the general statement

$$x(\Xi^{(k)}, m) = x(\Xi^{(k-1)}, m+1) = x(\Xi^{(k-2)}, m+2)$$

for $k, m = 0, 1, 2$. Differentiation of these relationships with respect to $\xi^{(0)}$ and $\eta^{(0)}$ and applying (21) yields

$$A_m(\Xi^{(k)})\, P^k = A_{m+r}(\Xi^{(k-r)})\, P^{k-r}$$

Simplifying,

$$A_m(\Xi^{(k)}) = A_{m+r}(\Xi^{(k-r)})\, P^{-r} \tag{22}$$

Now let $r = -m$ to obtain the result. §

4.4 Label-Invariance

Let $\alpha_m = det(A_m)$. From Proposition 2, it is immediate that

$$\alpha_m\,(\Xi^{(r)}) = \alpha_0\,(\Xi^{(r+m)}) \tag{23}$$

$$T_m\,(\Xi^{(r)})\, W\,(\Xi^{(r)}) = T_0\,(\Xi^{(r+m)})\, W\,(\Xi^{(r+m)})\, P^m \tag{24}$$

$$\tau_m\,(\Xi^{(r)})\, \omega\,(\Xi^{(r)}) = \tau_0\,(\Xi^{(r+m)})\, \omega\,(\Xi^{(r+m)}) \tag{25}$$

for any W. The relation $A_{m+1}\,(\Xi) = A_m\,(\Xi)\, P$ that held for the linear map does not hold for the quadratic map at arbitrary Ξ. As a consequence, the local metrics of interest are not label-invariant for arbitrary Ξ as they were in the linear case.

Proposition 3. Let $\Xi^c = \left(\frac{1}{3}, \frac{1}{3}\right)$. Then the metrics $\mu(T) = \tau$ and $\mu(T) = |T|$ are label-invariant at Ξ^c, the first for arbitrary reference element and the second for an equilateral element.

Proof. When $\Xi = \Xi^c$, the three symmetry points are all equal to Ξ^c. Then (25) becomes

$$\tau_m\,(\Xi^{(c)}) = \tau_0\,(\Xi^{(c)}) \tag{26}$$

Therefore, the metric $\mu(T) = \tau$ is label-invariant. Similarly, (24) becomes

$$T_m\,(\Xi^{(c)})\,W\,(\Xi^{(c)}) = T_0\,(\Xi^{(c)})\,W\,(\Xi^{(c)})\,P^m \tag{27}$$

Therefore,

$$|\,T_m\,(\Xi^{(c)})\,| = |\,T_0\,(\Xi^{(c)})\,W\,(\Xi^{(c)})\,P^m\,[\,W\,(\Xi^{(c)})\,]^{-1}\,| \tag{28}$$

But one can show that when the reference element is equilateral, $W(\Xi^c) \in \mathcal{M}$, so, as was shown in Proposition 1, $W\,P^m\,W^{-1}$ is a rotation. Using the Frobenius invariance property, we have

$$|\,T_m\,(\Xi^{(c)})\,| = |\,T_0\,(\Xi^{(c)})\,| \tag{29}$$

and thus the local metric $|T|$ is label-invariant. §

A similar proof can be constructed to show that $\mu(T) = |T^tT|$ is also label-invariant at Ξ^c, provided the reference element is equilateral.

Corollary. Metrics $\mu(T)$ that involve combinations of τ, $|T|$, and/or T^tT are label-invariant at $\Xi = \Xi^c$, provided $W(\Xi^c) \in \mathcal{M}$. For example, the Mean Ratio metric.

For the linear triangle map, the local metrics τ, $|T|$, and $|T^tT|$ were label-invariant for any Ξ because the Jacobian matrices were constant. Therefore, there was no need to consider symmetry metrics. Since the Jacobians vary with Ξ in the quadratic case, it is necessary to investigate symmetry metrics.

Proposition 4. If the local metric satisfies $\mu_{m+r}\,(\Xi^{(s)}) = \mu_m\,(\Xi^{(r+s)})$ for a particular reference element, then σ_m in (7) with $N = 3$ is label-invariant.

Proof

$$\sigma_{m+r} = \max\{\,\mu_{m+r}(\Xi^{(0)}),\,\mu_{m+r}(\Xi^{(1)}),\,\mu_{m+r}(\Xi^{(2)})\,\} \tag{30}$$

$$= \max\{\,\mu_m(\Xi^{(r)}),\,\mu_m(\Xi^{(r+1)}),\,\mu_m(\Xi^{(r+2)})\,\} \tag{31}$$

For any choice of r, we have $\sigma_{m+r} = \sigma_m$, and thus this symmetry metric is label-invariant. §

Corollary. When the local metric is $\mu(T) = \tau$ then σ_m is label-invariant, provided the reference triangle has straight sides.

Proof. When $k - r = s$, the relation (22) becomes

$$A^{(s)}_{m+r} = A^{(r+s)}_m\,P^r$$

from which one obtains

$$T^{(s)}_{m+r}\,W^{(s)} = T^{(r+s)}_m\,W^{(r+s)}\,P^r$$

and so

$$\tau_{m+r}^{(s)}\,\omega^{(s)} = \tau_m^{(r+s)}\,\omega^{(r+s)}$$

When the reference element has straight sides, $det(W)$ is constant, the previous becomes $\tau_{m+r}^{(s)} = \tau_m^{(r+s)}$ and therefore $\mu_{m+r}^{(s)} = \mu_m^{(r+s)}$. Thus the assumption of Proposition 4 is satisfied. §

Corollary. When the local metric is $\mu(T) = |T|$ then σ_m is label-invariant, provided the reference triangle is equilateral (with straight sides).

Proof. From the previous corollary

$$T_{m+r}\,(\Xi^{(s)}) = T_m\,(\Xi^{(r+s)})\,W^{(r+w)}\,P^r\,[\,W^{(s)}\,]^{-1} \tag{32}$$

When the reference element is equilateral, it has straight sides, and then W is independent of Ξ. Moreover, $W\,P^r\,W^{-1}$ is a rotation. Taking the norm of both sides of the above relation and using the rotation-invariance property of the Frobenius norm shows that the assumption of Proposition 4 is satisfied. §

One can similarly show that the symmetry metric σ_m based on the local metric $\mu(T) = |T^tT|$ is label-invariant provided the reference element is equilateral.

Corollary. When the local metric is $\mu(T) = |T|^2/2\tau$ then σ_m is label-invariant, provided the reference triangle is equilateral.

Proof. The previous corollaries showed $\tau_{m+r}^{(s)} = \tau_m^{(r+s)}$ and $|T_{m+r}(\Xi^{(s)})| = |T_m(\Xi^{(r+s)})|$. From this, one can readily see that the assumption of Proposition 4 is satisfied. §

More generally, any local metric that is a combination of τ and $|T|$ can be used to form a label-invariant symmetry metric provided the reference triangle is equilateral.

The local metric $\mu(T) = tr(T)$ is never label-invariant, nor is its associated symmetry metric label-invariant.

The results of Proposition 4 and its corollaries apply equally well to symmetry metrics based on the minimum (6) or the power mean (5) instead of the maximum (7).

If $\mu_{m+r}\,(\Xi^{(s)}) = \mu_m\,(\Xi^{(r+s)})$ for a particular choice of reference element, then for $s = 0$ we have $\mu_{m+r}\,(\Xi^{(0)}) = \mu_m\,(\Xi^{(r)})$. Then one can show that, for example, (7) becomes

$$\sigma_m = \max\{\,\mu_0\,(\Xi^{(0)}), \mu_1\,(\Xi^{(0)}), \mu_2\,(\Xi^{(0)})\} \tag{33}$$

This directly shows that the maximum-symmetry metric is label-invariant for a particular choice of reference element, and that one can evaluate it either by fixing the map and evaluating the local metrics at the three symmetry points, or, by varying the map and evaluating the local metric at the first symmetry point. The same is true for the minimum-symmetry and power-symmetry metrics.

4.5 The Shape Quality of Quadratic Triangles

As noted previously, [18] is the only reference which proposes quality metrics for quadratic elements. In theory one might also create an *element* metric based on some local metric μ as in the following example:

$$q_\varepsilon = \max_{\Xi \in U} \{ \mu(\Xi) \}$$

Unfortunately, the definition is impractical to compute efficiently due to the infinite number of points at which μ must be evaluated. As a practical alternative, consider using one or more sets of symmetry metrics. For example, in the quadratic triangle case let $\mathcal{S}_v$ consist of the symmetry points $\{ \Xi^{(0)}, \Xi^{(1)}, \Xi^{(2)} \}$ when $\Xi = (0,0)$, i.e., $\mathcal{S}_v = \{ (0,0), (1,0), (0,1) \}$. Likewise, let $\mathcal{S}_n = \{ \left(\frac{1}{2},\frac{1}{2}\right), \left(0,\frac{1}{2}\right) \left(\frac{1}{2},0\right) \}$, obtained when $\Xi = \left(\frac{1}{2},\frac{1}{2}\right)$. Now define three symmetry metrics

$$\sigma_v = \max_{\Xi \in \mathcal{S}_v} \{ \mu(\Xi) \} \tag{34}$$

$$\sigma_n = \max_{\Xi \in \mathcal{S}_n} \{ \mu(\Xi) \} \tag{35}$$

$$\sigma_{v+n} = \max_{\Xi \in (\mathcal{S}_v \cup \mathcal{S}_n)} \{ \mu(\Xi) \} \tag{36}$$

In words, the first symmetry metric evaluates the local metric at only the three corner vertices of the element, the second at only the three mid-side nodes, and the third on both sets.

To illustrate, the three metrics σ_v, σ_n, and σ_{v+n} were computed for ten thousand randomly generated quadratic triangles. The local metric μ was chosen to be the inverse mean ratio (shape) metric with an equilateral reference element. Each of these values was compared to the value q_ε, which was approximated by evaluating μ on 1250 points uniformly distributed over the logical triangle. Figure 4 compares the pairs $(\sigma_v, q_\varepsilon)$, $(\sigma_n, q_\varepsilon)$, and $(\sigma_{v+n}, q_\varepsilon)$ in three scatter plots whose range on both the x- and y-axes is -1 to 1 (1 being the best quality). Since for any triangle, $\sigma_v \le q_\varepsilon$ (and likewise for the other cases), the upper left side of each plot is empty. Sampling at only the mid-side nodes is the least effective of the three cases. As one can see, for some triangles even σ_{v+n} can be a poor approximation to q_ε.

On the positive side, it appears from the σ_{v+n} plot that the approximation to q_ε improves as the quality of the triangle improves. For example, there are relatively few points found below the $\sigma_{v+n} = q_\varepsilon$ line when the quality is better than 0.0, whereas there are a lot of points below the line when the quality is less than 0.0. That means for non-inverted elements, the approximation to element shape quality is not too bad in most instances. Further investigation is required in order to determine whether or this observation holds for other metrics and/or element types, but it is encouraging, at least. In any case, this third approach to measuring the quality of non-triangular quadratic elements is the only known approach.

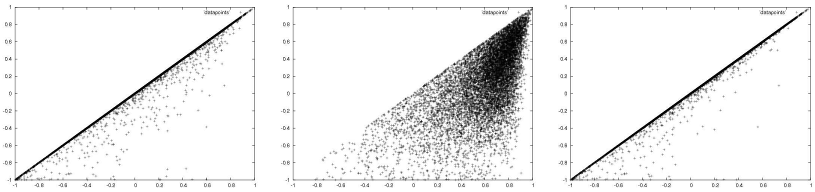

Fig. 4. Quadratic Planar Tri: σ_v, σ_n, σ_{v+n} vs. q_ε. (Left) Three Corner Vertices, (Middle) Three Mid-side Nodes, (Right) Six Vertices

Finally, we close this section with a proposition that applies to symmetry metrics for any map and further, makes it clear that the symmetry metric σ_{v+n} discussed in this section is label-invariant.

Proposition 5. Let $\mathcal{S}(\Xi) = \{\, \Xi^{(0)}, \Xi^{(1)}, \ldots, \Xi^{(N)} \,\}$ be a set of symmetry points. Let $\Xi_1 \in U$ and $\Xi_2 \in U$ be two values of Ξ, and further let $\mathcal{S}_1 = \mathcal{S}(\Xi_1)$ and $\mathcal{S}_2 = \mathcal{S}(\Xi_2)$. Let $\sigma_m(\Xi)$ be defined as one of (5)-(7). Suppose both $\sigma_m(\Xi_1)$ and $\sigma_m(\Xi_2)$ are label-invariant. Then σ_m evaluated on $\mathcal{S}_1 \cup \mathcal{S}_2$ is label-invariant.

Proof. The proof is constructed for the case where the symmetry metric is based on the maximum function. Similar proofs can be given for the other cases. Then

$$\sigma_m(\Xi_1) = \max_{\Xi^{(s)} \in \mathcal{S}_1} \{\, \mu_m(\Xi^{(s)}) \,\} \tag{37}$$

$$\sigma_m(\Xi_2) = \max_{\Xi^{(s)} \in \mathcal{S}_2} \{\, \mu_m(\Xi^{(s)}) \,\} \tag{38}$$

are label-invariant. Also define

$$\sigma_m(\Xi_1, \Xi_2) = \max_{\Xi^{(s)} \in (\mathcal{S}_1 \cup \mathcal{S}_2)} \{\, \mu_m(\Xi^{(s)}) \,\} \tag{39}$$

Therefore

$$\sigma_m(\Xi_1, \Xi_2) = \max\{\, \sigma_m(\Xi_1), \sigma_m(\Xi_2) \,\} \tag{40}$$

and

$$\begin{aligned} \sigma_{m+r}\,(\Xi_1, \Xi_2) &= \max\{\, \sigma_{m+r}\,(\Xi_1), \sigma_{m+r}\,(\Xi_2) \,\} \\ &= \max\{\, \sigma_m(\Xi_1), \sigma_m(\Xi_2) \,\} \\ &= \sigma_m(\Xi_1, \Xi_2) \end{aligned} \tag{41}$$

§

5 Summary

Quality measurement within mesh elements can be achieved using local metrics such as those given in the Target-matrix paradigm, along with a map from the logical to the reference and physical elements. Local area, volume, shape, and orientation can thus be measured with respect to the same local quantities within the reference element. This provides a method for assessing the quality of elements having curved sides, such as those associated with the quadratic map. The minimum or maximum value of these quantities over all points in the element can be approximated by taking local measurements on a small, finite, set of points. For elements whose quality is not too bad (e.g., non-inverted), the approximations appear reasonably good, as seen in the quadratic triangle example. In any case, this 'hybrid' quality method is the only known approach to measuring the quality of non-triangular high-order elements.

Label-invariance is a desirable property for some quality metrics. Local metrics can be made label invariant by evaluating them at the center of the element and using a particular reference element. For shape metrics, the appropriate reference element for label-invariance was the regular shape corresponding to the given element type. For size metrics, the reference element was arbitrary. Metrics that are sensitive to orientation, such as those involving the trace, are not label-invariant for any choice of reference element.

Another type of label-invariance can be obtained by defining a symmetry metric, based on an associated local metric and a set of symmetry points that differs from one element type to another. As in the local metric case, the symmetry metrics can be made label invariant provided the reference element is regular. An advantage of the symmetry metrics is that one does not have to evaluate the local metric at the center of the element in order to obtain label-invariance. This is important because, for example, quality at the corners can often provide a more discerning criterion than quality at the center.

Similar results for the linear and quadratic tetrahedron, and the linear planar quadrilateral, are given in [13]. It is expected the similar results would hold for the quadratic quadrilateral and for the linear and quadratic hexahedral elements. Pyramid and prismatic elements are not naturally isotropic, so probably local metrics on these cannot be made label-invariant. Non-planar quadrilaterals and non-planar quadratic triangles have not been investigated for label-invariance.

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