Triangulation of Simple 3D Shapes with Well-Centered Tetrahedra

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Summary. A completely well-centered tetrahedral mesh is a triangulation of a three dimensional domain in which every tetrahedron and every triangle contains its circumcenter in its interior. Such meshes have applications in scientific computing and other fields. We show how to triangulate simple domains using completely well-centered tetrahedra. The domains we consider here are space, infinite slab, infinite rectangular prism, cube, and regular tetrahedron. We also demonstrate single tetrahedra with various combinations of the properties of dihedral acuteness, 2-well-centeredness, and 3-well-centeredness.

1 Introduction

In this paper we demonstrate well-centered triangulation of simple domains in $\mathbb{R}^3$. A well-centered simplex is one for which the circumcenter lies in the interior of the simplex (8). This definition is further refined to that of a k-well-centered simplex which is one whose k-dimensional faces have the well-centeredness property. An n-dimensional simplex which is k-well-centered for all $1 \leq k \leq n$ is called completely well-centered (15). These properties extend to simplicial complexes, i.e. to meshes. Thus a mesh can be completely well-centered or k-well-centered if all its simplices have that property. For triangles, being well-centered is the same as being acute-angled. But a tetrahedron can be dihedral acute without being 3-well-centered as we show by example in Sect. 2. We also note that while every well-centered triangulation is Delaunay, the converse is not true.

In (14) we described an optimization-based approach to transform a given planar triangle mesh into a well-centered one by moving the internal vertices while keeping the boundary vertices fixed. In (15) we generalized this approach to arbitrary dimensions and in addition to developing some theoretical properties of our method showed some complex examples of our method at work in the plane and some simple examples in $\mathbb{R}^3$. In this paper the domains we consider are space, slabs, infinite rectangular prisms, cubes and tetrahedra.

Well-centered triangulations find applications in some areas of scientific computing and other fields. Their limited use so far may be partly because until recently there were no methods known for constructing such meshes. One motivation for constructing well-centered meshes comes from Discrete Exterior Calculus, which is a framework for constructing numerical methods for partial differential equations (4, 8). The availability of well-centered meshes permits one to discretize an important operator called the Hodge star as a diagonal matrix, leading to efficiencies in numerical solution procedures. Other potential applications are the covolume method (10, 11), space-time meshing (13), and computations of geodesic paths on manifolds (9).

2 Well-Centeredness and Dihedral Acuteness for a Single Tetrahedron

An equatorial ball of a simplex is a ball for which the circumsphere of the simplex is an equator. Stated more precisely, if σ^k is a k-dimensional simplex, then the equatorial ball of σ^k is a $(k+1)$-dimensional ball whose center is $c(\sigma^k)$, the circumcenter of σ^k, and whose radius is $R(\sigma^k)$, the circumradius of σ^k. In (15) we showed that a simplex σ^n is n-well-centered if and only if for each vertex v in σ^n, v lies outside the equatorial ball of the facet τ_v^{n-1} opposite v. Figure 1 illustrates this characterization of well-centeredness; the vertex v at the top of the tetrahedron in Fig. 1 lies outside of the equatorial ball of the facet opposite v.

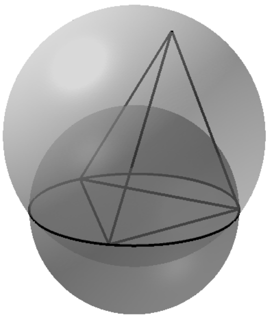


Fig. 1. One characterization of well-centeredness is that for each vertex v, the vertex lies outside of the equatorial ball of the facet opposite v. The larger sphere shown here is the circumsphere of the tetrahedron and the smaller sphere is the boundary of the equatorial ball of the bottom triangle. The top vertex of the tetrahedron is outside the equatorial ball.

We now have a definition of k-well-centeredness and an alternate characterization of well-centeredness, but these provide limited intuition for what it means to be well-centered. In this section we discuss a variety of tetrahedra, showing that in $\mathbb{R}^3$, a simplex that is 2-well-centered may or may not be 3-well-centered, and vice versa. We also discuss how being dihedral acute relates to being well-centered.

Figures 2 through 7 are pictures of six different tetrahedra that illustrate the possible combinations of the qualities 2-well-centered, 3-well-centered, and dihedral acute. Each picture shows a tetrahedron inside of its circumsphere. The center of each circumsphere is marked by a small, unlabeled axes indicator. In each case, the circumcenter of the tetrahedron lies at the origin, and the circumradius is 1. The coordinates given are exact, and the quality statistics are rounded to the nearest value of the precision shown.

The quality statistics displayed include the minimum and maximum face angles and dihedral angles of the tetrahedron. These familiar quality measures need no further explanation, and it is easy to determine from them whether a tetrahedron is 2-well-centered (having all face angles acute) or dihedral acute. The R/ℓ statistic shows the ratio of the circumradius R to the shortest edge of the tetrahedron, which has length ℓ. The range of R/ℓ is $[\sqrt{3/8}, \infty]$, with $\sqrt{3/8} \approx 0.612$. A single tetrahedron has a particular R/ℓ ratio, so the minimum R/ℓ equals the maximum R/ℓ in each of Figs. 2–7. Later, however, we will show similar statistics for tetrahedral meshes, and it is convenient to use the same format to summarize mesh quality in both cases. The R/ℓ ratio is a familiar measurement of the quality of a tetrahedron, especially in the context of Delaunay refinement.

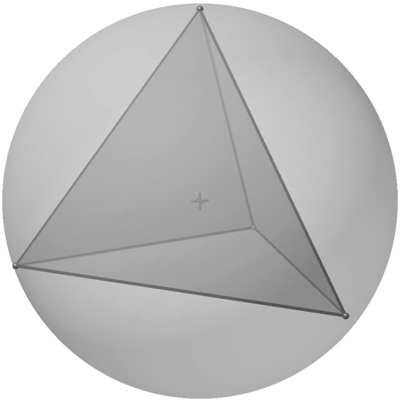
The quality statistic h/R is less familiar. The R in this ratio is the circumradius. The h stands for height. For a given facet of the tetrahedron, h measures the signed height of the circumcenter of the tetrahedron above the plane containing that facet. The direction above the facet means the direction towards the remaining vertex of the tetrahedron, and h is positive when the circumcenter lies above the facet. In (15) the quantity h/R and its relationship to well-centeredness in any dimension is discussed at more length. For our purposes it should suffice to note that the range of h/R for tetrahedra is $(-1, 1)$, that a tetrahedron is 3-well-centered if and only if the minimum h/R is positive, and that $h/R = 1/3$ relative to every facet of the regular tetrahedron.

The regular tetrahedron is completely well-centered and dihedral acute. Our first example, shown in Fig. 2, is another tetrahedron that shares those properties with the regular tetrahedron. Not every completely well-centered tetrahedron is dihedral acute, and the second example, shown in Fig. 3, is a tetrahedron that is completely well-centered but not dihedral acute. Some sliver tetrahedra are completely well-centered and have dihedral angles approaching $180°$.

As one might expect, there are tetrahedra that have none of these nice properties. The tetrahedron shown in Fig. 4 is an example of such a tetrahedron. Most polar caps also have none of these nice properties. Some tetrahedra that are neither dihedral acute nor 2-well-centered nor 3-well-centered have much worse quality than the example in Fig. 4. This particular example is near the boundary dividing 3-well-centered tetrahedra from tetrahedra that are not 3-well-centered.

The tetrahedra we have considered so far have been either completely well-centered or neither 2-well-centered nor 3-well-centered. Our last three examples show that tetrahedra can be 2-well-centered without being 3-well-centered and vice-versa. The tetrahedron shown in Fig. 5 is 2-well-centered and dihedral acute, but not 3-well-centered. The example tetrahedron shown in Fig. 6 is similar but has been modified to no longer be dihedral acute. Our final example, shown in Fig. 7, is a tetrahedron that is 3-well-centered, but is neither 2-well-centered nor dihedral acute.

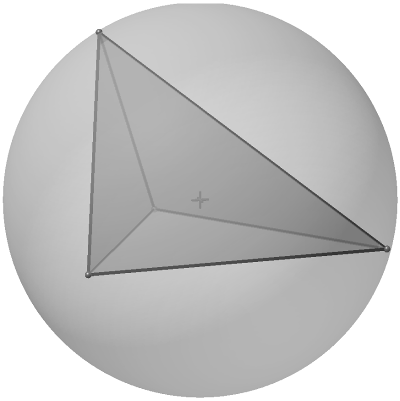
Table 1 summarizes the six examples presented in this section, indicating whether each example is 2-well-centered, 3-well-centered, and/or dihedral acute. Of the eight possible binary sequences, the two missing examples are the sequences N, N, Y and Y, N, Y, in which a tetrahedron would be dihedral acute but not 2-well-centered. These



Vertex Coordinates		
x	y	z
0.6	−0.64	−0.48
0.48	0.8	−0.36
−0.96	0	−0.28
0	0	1

Quality Statistics		
Quantity	Min	Max
h/R	0.254	0.371
Face Angle	50.92°	67.08°
Dihedral Angle	58.76°	76.98°
R/ℓ	0.690	0.690

Fig. 2. A tetrahedron that is completely well-centered and dihedral acute



Vertex Coordinates		
x	y	z
0	0.96	−0.28
−0.744	−0.64	−0.192
0.856	−0.48	−0.192
−0.48	0.192	0.856

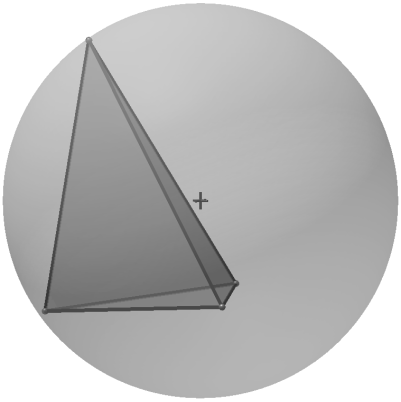
Quality Statistics		
Quantity	Min	Max
h/R	0.224	0.427
Face Angle	46.26°	77.62°
Dihedral Angle	52.71°	94.15°
R/ℓ	0.733	0.733

Fig. 3. A completely well-centered tetrahedron that is not dihedral acute

Vertex Coordinates		
x	y	z
0.224	−0.768	−0.6
0.8	0	−0.6
0.224	0.768	−0.6
−0.28	0	0.96

Quality Statistics		
Quantity	Min	Max
h/R	−0.029	0.600
Face Angle	29.89°	106.26°
Dihedral Angle	35.42°	116.68°
R/ℓ	1.042	1.042

Fig. 4. A tetrahedron that is not dihedral acute, 2-well-centered, or 3-well-centered



Vertex Coordinates		
x	y	z
0.36	-0.8	-0.48
0.768	0.28	-0.576
-0.6	0.64	-0.48
0.576	0.168	0.8

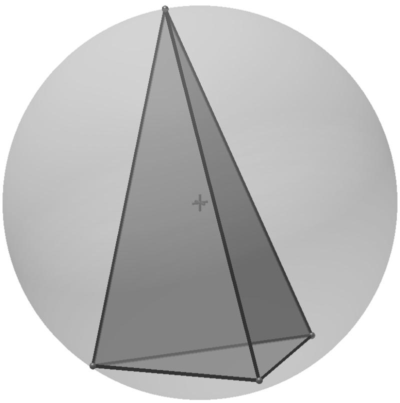
Quality Statistics		
Quantity	Min	Max
h/R	-0.109	0.562
Face Angle	$41.71°$	$83.76°$
Dihedral Angle	$53.33°$	$85.72°$
R/ℓ	0.863	0.863

Fig. 5. A tetrahedron that is dihedral acute and 2-well-centered, but not 3-well-centered

Vertex Coordinates		
x	y	z
-0.152	0.864	-0.48
-0.64	-0.6	-0.48
0.6	-0.64	-0.48
-0.192	-0.64	0.744

Quality Statistics		
Quantity	Min	Max
h/R	-0.024	0.630
Face Angle	$42.08°$	$85.44°$
Dihedral Angle	$59.94°$	$91.20°$
R/ℓ	0.806	0.806

Fig. 6. A tetrahedron that is not dihedral acute or 3-well-centered, but is 2-well-centered



Vertex Coordinates		
x	y	z
0	-0.6	-0.8
0.64	-0.024	-0.768
-0.64	-0.024	-0.768
0	0.352	0.936

Quality Statistics		
Quantity	Min	Max
h/R	0.112	0.765
Face Angle	$25.69°$	$95.94°$
Dihedral Angle	$40.33°$	$105.62°$
R/ℓ	1.161	1.161

Fig. 7. A tetrahedron that is not dihedral acute or 2-well-centered, but is 3-well-centered

Table 1. A tetrahedron may have any of six different possible combinations of the qualities 2-well-centered, 3-well-centered, and dihedral acute

Figure	3-WC	2-WC	Acute
2	Y	Y	Y
3	Y	Y	N
4	N	N	N
5	N	Y	Y
6	N	Y	N
7	Y	N	N

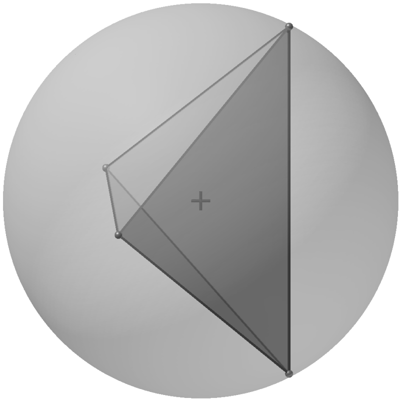
examples are missing because they do not exist; every tetrahedron that is dihedral acute is also 2-well-centered. Eppstein, Sullivan, and Üngör provide a proof of this in Lemma 2 of (6), which states, among other things, that "an acute tetrahedron has acute facets."

3 Tiling Space, Slabs, and Infinite Rectangular Prisms with Completely Well-Centered Tetrahedra

We have mentioned above that there are applications that could make good use of well-centered triangulations and have considered examples of single well-centered tetrahedra. In what follows, we give some examples of well-centered triangulations of simple domains in dimension 3.

In (6), Eppstein, Sullivan, and Üngör show that one can tile space, $\mathbb{R}^3$, and infinite slabs, $\mathbb{R}^2 \times [0, a]$, with dihedral acute tetrahedra. They also briefly discuss how high-quality tilings of space have been used to design meshing algorithms. The acute triangulations of space given in (6) all make use of copies of at least two different tetrahedra, and the authors suggest it is unlikely that there is a tiling of space with copies of a single acute tetrahedron. Their acute triangulation of the slab appears to use copies of seven distinct tetrahedra. The problem of triangulating an infinite rectangular prism, $\mathbb{R} \times [0, a] \times [0, b]$, or a cube, $[0, 1]^3$, with acute tetrahedra is still an open problem as far as the authors know.

In contrast to the complexity of tiling space with acute tetrahedra, there are fairly simple completely well-centered triangulations of space. Barnes and Sloane proved that the optimal lattice for quantizing uniformly distributed data in $\mathbb{R}^3$ is the body-centered cubic (BCC) lattice (2). Since this is related to centroidal Voronoi tesselations (CVTs) (5), and CVTs have been used for high-quality meshing of 3-dimensional domains (see (1)), it is not surprising that a Delaunay triangulation of the vertices of the BCC lattice gives rise to a high quality triangulation of space. The triangulation consists of congruent copies of a single completely well-centered tetrahedron, shown in Fig. 8. The tiling is one of four spatial tilings discovered by Sommerville (6, 12). The other three tilings Sommerville found are neither 3-well-centered nor 2-well-centered, though one of them has a maximum face angle of $90°$ and is dihedral nonobtuse. It is interesting to note that Fuchs algorithm for meshing spatial domains based on high-quality spatial tilings had "good performance ... when he used the second Sommerville construction," S(6, 7) which is the completely well-centered tetrahedron shown in Fig. 8.



Vertex Coordinates		
x	y	z
-1	-2	0
1	0	-2
-1	2	0
1	0	2

Quality Statistics		
Quantity	Min	Max
h/R	0.316	0.316
Face Angle	$54.74°$	$70.53°$
Dihedral Angle	$60.00°$	$90.00°$
R/ℓ	0.645	0.645

Fig. 8. The second Sommerville tetrahedron – a completely well-centered tetrahedron that tiles space

Next we describe the space tiling that uses the Sommerville tetrahedron described above. The view of the tetrahedron shown in Fig. 8 has an elevation angle between 10 and 11 degrees, so it is not difficult to identify the horizontal edge and the vertical edge of the tetrahedron. The horizontal edge connects a pair of vertices from one of the cubic lattices, and the vertical edge connects a pair of vertices from the other cubic lattice. It is natural to think of this spatial tiling in terms of the interleaved cubic lattices, but we can also look at the tiling from a different perspective. Consider the tilted plane P that contains the bottom face of the tetrahedron shown in Fig. 8. Consider a single cube of one of the cubic lattices. The plane P is the one defined by two opposing parallel edges of the cube. At the center of the cube there is a vertex from the other cube lattice; this vertex also lies in P. By making translated copies of the vertices in P we can obtain all of the vertices of the BCC lattice. Each vertex in P has six adjacent vertices in P and four adjacent vertices in each of the plane above and below P. There are two types of tetrahedra in the tiling, but both types are copies of the same tetrahedron in this case. The first type is the convex hull of three vertices in a copy of P and one vertex from the plane above or below that copy of P. The second type of tetrahedron is the convex hull of an edge in a copy of P and a corresponding edge from the plane above or below that copy of P.

From this understanding of the structure of the BCC-based spatial tiling, we can generalize to an entire family of triangulations of space using copies of two different tetrahedra. We consider first the set of vertices $\{(i, 0, 0) : i \in \mathbb{Z}\}$. We will make translated copies of this line in one direction to make a plane P. To do this we choose a parameter $a > 0$ and make infinitely many copies of each vertex translating by the vector $(1/2, a, 0)$. Lastly we choose a parameter $b > 0$ and make translated copies of the plane P using the translation vector $(1/2, 0, b)$. Thus our set of vertices is

$$\{(i + j/2 + k/2, aj, bk) : i, j, k \in \mathbb{Z}\},$$

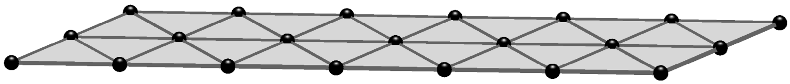


Fig. 9. Starting with a set of vertices equally spaced along a line, we make translated copies of the line in a plane and triangulate the plane with each vertex having six neighbors

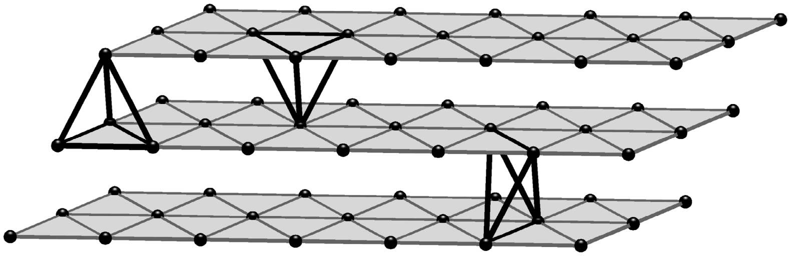


Fig. 10. We make translated copies of the plane into space and form tetrahedra by connecting vertices of adjacent planes as shown

i.e., the lattice $\Lambda = \{\sum_{i=1}^{3} c_i u_i : c_i \in \mathbb{Z}\}$ with basis vectors

$$u_1 = (1, 0, 0), u_2 = (1/2, a, 0), \text{ and } u_3 = (1/2, 0, b).$$

To turn this into a triangulation of space, we start by triangulating each copy of the plane P as shown in Fig. 9. Each vertex v in a copy of P is connected to the six vertices which lie at positions $v \pm (1, 0, 0)$, $v \pm (1/2, a, 0)$, and $v \pm (1/2, -a, 0)$. This yields the standard tiling of the plane with equilateral triangles if $a = \sqrt{3}/2$. Now for each triangle in a copy of P there is exactly one edge – the edge in the direction $(1, 0, 0)$ – for which a vertex lies directly above and directly below the midpoint of that edge. The first type of tetrahedron, then, is the convex hull of a triangle T in P and the vertex directly above or below the midpoint of one of the edges of T. If we add all possible tetrahedra of the first type, the gaps that remain are all tetrahedra of the second type. This second type of tetrahedron is the convex hull of an edge in P in the direction $\pm(1/2, a, 0)$ and the edge in the direction $\pm(1/2, -a, 0)$ whose midpoint lies directly above or below the midpoint of the given edge. Figure 10 shows three copies of the plane and highlights three of the tetrahedra that appear in the spatial tiling. The two tetrahedra on the left side of Fig. 10 are both copies of the first type of tetrahedron. The other tetrahedron is of the second type. For the BCC-based spatial triangulation, in which both types of tetrahedra are the same, the parameters are $a = b = \sqrt{2}/2$.

This family of triangulations of space is interesting for at least two reasons. First, the triangulation is completely well-centered if and only if both $a > 1/2$ and $b > 1/2$. Second, the family provides an elegant solution to the problems of tiling an infinite slab in $\mathbb{R}^3$ and tiling infinite rectangular prisms in $\mathbb{R}^3$. To tile the slab, one uses a finite number of translates of plane P. We see that any slab can be triangulated using copies of a single tetrahedron; for the parameters $a = b = \sqrt{2}/2$, the two types of tetrahedra are the same, and we can scale the result as needed to get a slab of the desired thickness. Triangulating rectangular prisms is also easy; it suffices to use a finite number of translates both of the initial line and of the resulting infinite strip. Again, this can be done using copies of a single tetrahedron, provided that the ratio of side lengths of the rectangle is a rational number. If the ratio of side lengths is p/q, one can use parameters $a = b = \sqrt{2}/2$ and take p copies of the initial line with q copies of the infinite strip. The result has the correct ratio of side lengths and can be scaled to get the desired rectangle.

4 Meshing the Cube With Well-Centered Tetrahedra

Meshing the cube with well-centered tetrahedra is significantly more difficult than meshing an infinite square prism, but it can be done. In this section we discuss several well-centered meshes of the cube that we constructed. In each case, the mesh was built by first designing the mesh connectivity, then moving the internal vertices using the optimization algorithm described in (15).

The first completely well-centered mesh of the cube that we discovered has 224 tetrahedra. We do not discuss the construction of this mesh in detail here, but it is worth mentioning the mesh because it has higher quality than the other well-centered meshes of the cube we will discuss. The quality statistics for the mesh are shown in the columns to the left in Table 2. The faces of this triangulation of the cube match up with each other, so it is relatively easy to make a well-centered mesh of any figure that can be tiled with unit cubes. One can use a copy of this well-centered mesh of the cube in each cube tile, using rotations and reflections as needed to make all the faces match.

Cassidy and Lord showed that the smallest acute triangulation of the square consists of eight triangles (3). Knowing that a completely well-centered mesh of the cube exists, it is natural to ask the analogous question for the cube. What is the smallest well-centered mesh of the cube? One can ask this question for the three different types of well-centered tetrahedral meshes – 2-well-centered, 3-well-centered, and completely well-centered. It is also conceivable that the well-centered mesh of the cube with the fewest tetrahedra is different from the mesh of the cube with the fewest vertices or edges, but we restrict our attention to the smallest well-centered mesh in the sense of fewest tetrahedra.

The answer to the question is not known for any of the three types of well-centered triangulations, but we can give some upper and lower bounds. The best known upper bound for 2-well-centered and completely well-centered triangulations of the cube is a completely well-centered mesh of the cube with 194 tetrahedra. Figure 11 shows a picture of this mesh. The quality statistics for the mesh are recorded in the middle columns of Table 2.

It is possible to improve the quality slightly by optimization of the location of the surface vertices, but for the version of the mesh shown in Fig. 11, the surface triangulation has some desirable symmetries, with every surface vertex at a cube corner, at the midpoint of an edge of the cube, or on a diagonal of a face of the cube. All of the faces match each other up to rotation and reflection, so this triangulation of the cube can also be used to create a well-centered mesh of any figure that can be tiled with unit cubes. Although the surface triangulation of this mesh is combinatorially the same as the triangulation with 224 tetrahedra, the vertices on the surfaces are not in the same location,

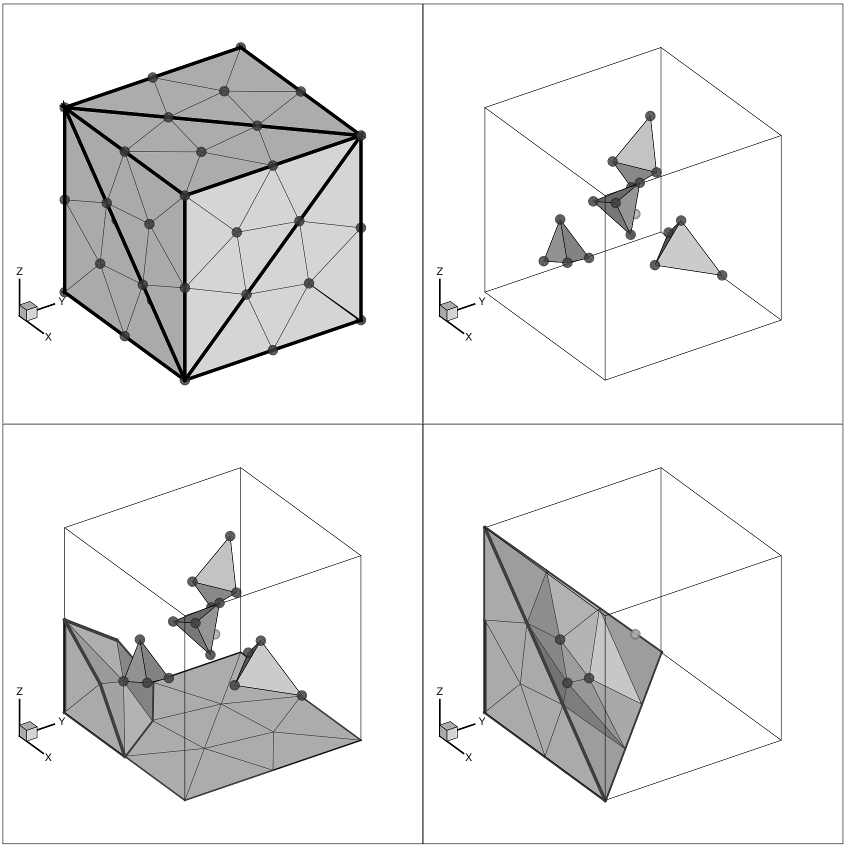


Fig. 11. A completely well-centered mesh of the cube with 194 tetrahedra

so it is not trivial to mix these two triangulations in meshing a cube-tiled shape. It is worth noting that the vertices of the surface triangulation that lie on the cube diagonals do not have coordinates of $1/3$ or $2/3$. Instead the coordinates are 0.35 or 0.65 for the vertices adjacent to an edge through the center of the cube face, and the coordinates are 0.295 or 0.705 for vertices not adjacent to such an edge.

Figure 11 is designed to make it possible to discern the structure of this triangulation of the cube. In the bottom right we see a triangulation of a region that fits into the corner of the cube and extends along the cube surface to diagonals of three of the faces of the cube. The triangulation basically has two layers of tetrahedra. The first layer is shown at bottom left. It consists of six tetrahedra that are incident to the corner of the cube. The two-layer triangulation is imprecisely replicated in four different corners of the cube. The thicker diagonal lines in the picture at top left help show which cube corners contain this type of triangulation. To complete the triangulation of the cube, a vertex at the center of the cube is added to the triangulations of the cube corners. The picture at

Table 2. The quality of our meshes of the cube decreases with the number of tetrahedra in the mesh of the cube

Cube Mesh Quality Statistics						
Quantity	224 Tets		194 Tets		146 Tets	
	Min	Max	Min	Max	Min	Max
h/R	0.041	0.850	0.005	0.790	0.016	0.854
Face Angle	21.01°	87.49°	26.93°	89.61°	17.09°	112.60°
Dihedral Angle	24.91°	105.61°	28.26°	126.64°	10.73°	163.17°
R/ℓ	0.618	1.569	0.612	1.134	0.711	1.835

top right of Fig. 11 shows all the interior vertices of the cube, combining the vertex at the center of the cube with one tetrahedron from each triangulation of a cube corner. The Delaunay triangulation defines the connectivity table, since any 3-well-centered triangulation is Delaunay.

This mesh establishes upper bounds for the 2-well-centered and completely well-centered cases. There is an even smaller triangulation of the cube that is known to be 3-well-centered. We do not describe it here except to say that the mesh consists of 146 tetrahedra and has a surface triangulation with fewer triangles than the two previously mentioned triangulations of the cube. The quality of that 3-well-centered mesh of the cube is shown in the rightmost columns of Table 2.

The upper bounds for this problem are simple to present, since they consist of constructive examples. The analytical lower bounds are rather more complicated to explain, and we discuss them here only briefly. One can show that in a 3-well-centered triangulation of the cube, no face of the cube is triangulated with two right triangles meeting along the hypotenuse (16). Thus each face of the cube must contain at least five vertices and at least three tetrahedral facets. Since there are six cube faces, this leads to a count of 18 tetrahedra, one adjacent to each facet. Some of these tetrahedra may have been counted twice, though, since a single tetrahedron may have a facet in each of two different cube faces. It can be shown that no tetrahedron having a facet in each of three different cube faces is a 3-well-centered tetrahedron (16). Thus no tetrahedron is triple-counted. A lower bound of 9 tetrahedra follows. It is possible to prove that any triangulation of the cube with all vertices lying on the edges of the cube is not 3-well-centered, but it is not immediately clear whether this can be used to improve the lower bound.

The lower bound for 2-well-centered and completely well-centered triangulations of the cube is slightly better. In this case, each face of the cube must be triangulated with an acute triangulation, so each cube face contains at least 8 tetrahedral facets. This gives a count of 48 tetrahedra, and once again we cannot have counted any tetrahedron three times (16). (For the 2-well-centered case this is not exactly the same reason as the 3-well-centered case.) A lower bound of 24 tetrahedra follows. This lower bound can be improved a little by paying attention to which triangular facets can and cannot lead to double-counted tetrahedra, and a lower bound of 30 tetrahedra can be obtained.

A careful analysis along these lines might improve the lower bound even more, since there is no way to conformally triangulate all the surfaces of the cube with an 8-triangle acute triangulation of each face. In any case, the authors suspect that the actual answer to these questions is close to 100 tetrahedra, if not greater, so the lower bounds mentioned here should be considered preliminary.

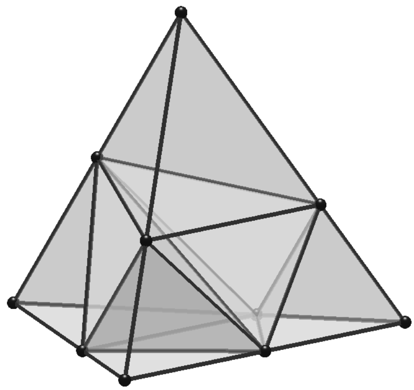
5 Some Subdivisions of Tetrahedra into Well-Centered Tetrahedra

Having seen that the cube can be subdivided into well-centered tetrahedra, one might also ask whether a tetrahedron can be subdivided into well-centered tetrahedra. Subdivisions of tetrahedra into well-centered tetrahedra could be used to refine an existing mesh. Subdividing a tetrahedron that tiles space into well-centered tetrahedra would provide new well-centered meshes of space. In general, well-centered subdivisions of tetrahedra might be used to design high-quality meshing algorithms.

One might suppose that subdividing the regular tetrahedron into smaller well-centered tetrahedra is relatively simple, but the problem is not so easy as it might seem. In two dimensions, the Loop subdivision, which refines a triangle by connecting the midpoints of each edge of the triangle, produces four smaller triangles. Each of these triangles is similar to the original triangle, so the Loop subdivision of an acute triangle is an acute triangulation of the triangle. In three dimensions, however, there is no obvious analog of the Loop subdivision.

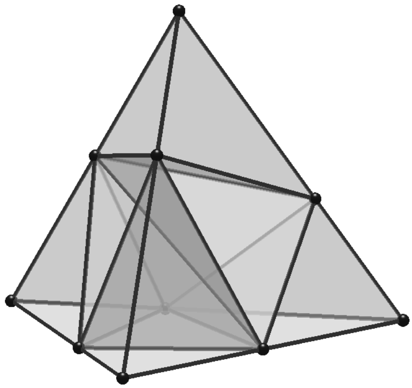
Connecting the midpoints of the edges of a tetrahedron cuts out four corner tetrahedra that are similar to the original tetrahedron. The shape that remains in the center after removing these four tetrahedra is an octahedron. In the case of the regular tetrahedron, it is a regular octahedron, and it can be subdivided into four tetrahedra by adding an edge between opposite vertices of the octahedron. The result is not well-centered; the center of the octahedron is the circumcenter of all four tetrahedra, so the tetrahedra are not 3-well-centered. In addition, the facets incident to the new edge are right triangles, so the tetrahedra are not 2-well-centered.

We can turn this subdivision of the regular tetrahedron into a well-centered subdivision, however. By sliding some of the new vertices along the edges of the regular



Quality Statistics		
Quantity	Min	Max
h/R	0.0345	0.712
Face Angle	38.87°	86.76°
Dihedral Angle	38.44°	121.37°
R/ℓ	0.702	0.934

Fig. 12. A simple subdivision of the regular tetrahedron into eight completely well-centered tetrahedra



Quality Statistics		
Quantity	Min	Max
h/R	0.0448	0.584
Face Angle	39.63°	87.43°
Dihedral Angle	46.72°	105.95°
R/ℓ	0.777	0.826

Fig. 13. Another simple subdividision of the regular tetrahedron into eight completely well-centered tetrahedra

tetrahedron, moving them away from the edge midpoints, we can make the four center tetrahedra well-centered without degrading the four corner tetrahedra to the point of not being well-centered. Figures 12 and 13 illustrate two different successful ways we can slide the vertices along edges of the tetrahedra. In both cases, the midpoint vertices that are adjacent to the central edge remain stationary, to keep the central edge as short as possible. In Fig. 12, the four free midpoints all slide towards the same edge of the regular tetrahedron. In Fig. 13, the free midpoints slide along a directed four-cycle through the vertices of the regular tetrahedron.

There are also more complicated ways to divide the regular tetrahedron into smaller well-centered tetrahedra. Figure 14 shows the basic structure of a subdivision of the regular tetrahedron into 49 tetrahedra. A smaller regular tetrahedron is placed in the center of the large tetrahedron with the same orientation as the original. Each face of the smaller tetrahedron is connected to the center of a face of the larger tetrahedron. At each corner of the large tetrahedron, a small regular tetrahedron is cut off of the corner, and the resulting face is connected to a vertex of the central regular tetrahedron. After filling in a few more tetrahedral faces, six more edges need to be added to subdivide octahedral gaps into tetrahedra. The mesh shown in Fig. 15 is the completely well-centered mesh that results from optimizing the mesh shown in Fig. 14. This subdivision of the regular tetrahedron is interesting partly because all of the surface triangulations match and have three-fold radial symmetry. It is also possible that this type of subdivision will be easier to use in mesh refinement than the other two subdivisions.

It is not clear whether these constructions can be extended in some way to create a well-centered subdivision of any well-centered tetrahedron. The constructions cannot be extended to create well-centered subdivisions of all tetrahedra, since both constructions cut off the corners of the tetrahedron to create smaller tetrahedra that are nearly similar to the original tetrahedron. In particular, the cube corner tetrahedron, i.e., some scaled, rotated, translated version of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, cannot be subdivided into well-centered tetrahedra in this fashion; one can show that no tetrahedron with three mutually orthogonal faces is

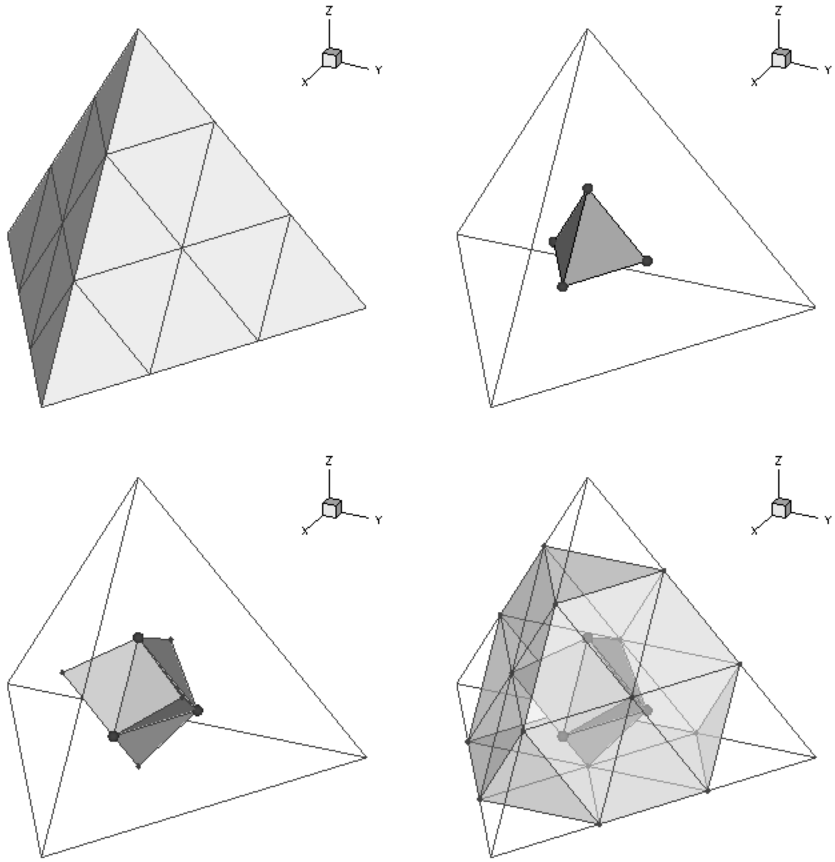
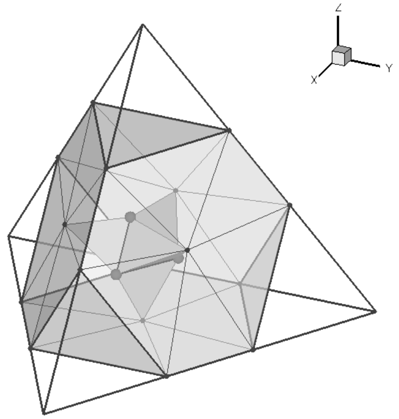


Fig. 14. A subdivision of the regular tetrahedron that can be made completely well-centered through optimization

3-well-centered (16). Subdividing the cube corner tetrahedron is particularly interesting, though, because it provides some guidance regarding what is needed to mesh a cube or, for that matter, any object having three mutually orthogonal faces that meet at a point.

In fact, the smallest known subdivision of the cube into well-centered tetrahedra is based on a subdivision of the cube corner into well-centered tetrahedra. The picture in the bottom right corner of Fig. 11 shows a triangulation of a region that fits into the corner of a cube. The three visible interior vertices in Fig. 11 are not coplanar with the cube diagonals, but they are nearly so, and there is a completely well-centered mesh of the cube corner that is combinatorially the same as that mesh. We actually obtained the mesh of the cube corner tetrahedron first, and the well-centered mesh of the cube in



Quality Statistics		
Quantity	Min	Max
h/R	0.0146	0.845
Face Angle	23.36°	89.07°
Dihedral Angle	29.93°	107.73°
R/ℓ	0.612	1.305

Fig. 15. The completely well-centered subdivision of the regular tetrahedron that results from optimizing the mesh shown in Fig. 14

Fig. 11 was obtained by replicating the cube corner mesh as described earlier, adding a vertex at the cube center, computing the Delaunay triangulation, and optimizing with the software discussed in (15).

6 Conclusions and Questions

We have discussed some of the properties of well-centered tetrahedra and seen that it is possible to triangulate a variety of basic three-dimensional shapes with completely well-centered tetrahedra. The triangulations discussed suggest that it might soon be practical to mesh simple domains in $\mathbb{R}^3$ with well-centered tetrahedra. They also show that there are a rich variety of well-centered tetrahedra.

The authors hope that it will be possible to build robust software for meshing three-dimensional domains with this variety of well-centered tetrahedra, but there is still significant work to be done before that goal can be reached. Part of this work is to determine what properties the neighborhood of a vertex in a 3-dimensional triangulation must have in order to permit a well-centered triangulation. The question is partially answered in (16), but a complete answer is lacking. The ability to subdivide any tetrahedron (or even just any well-centered tetrahedon) into smaller well-centered tetrahedra would also be a significant advance toward this goal.

This work also raises some questions that are of more theoretical interest, though not without practical application. It would be interesting to construct the smallest possible well-centered mesh of the cube. How might one improve the lower bounds on the number of tetrahedra needed in a well-centered mesh of the cube? Also it is still an open question whether there are other tetrahedra for which copies of a single tetrahedron meet face to face and fill space. Could there be other well-centered tetrahedra that tile space? Are there other families of high-quality tetrahedra that tile space?

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