Automated Quadrilateral Coarsening by Ring Collapse

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Summary. A uniform finite element mesh rarely provides the best discretization of a domain to accommodate a solution with both optimal efficiency and minimal error. Mesh adaptation can approach a more optimal solution by accommodating regions of the mesh with higher or lower element density. Extensive attention has been given to mesh adaptation in both computational mechanics and computer graphics to provide or improve methods for increasing the model resolution or solution accuracy. The algorithm developed in this paper, entitled Automated Quadrilateral Coarsening by Ring Collapse (AQCRC), provides a unique solution to allow mesh coarsening of both structured and unstructured quadrilateral meshes. The algorithm is based on modification and removal operations utilizing the dual description of the quadrilateral mesh. The AQCRC algorithm iterates on five steps: 1) input of a coarsening region and a coarsening factor, 2) selection of coarsening rings, 3) mesh quality improvement, 4) removal of coarsening rings, and 5) mesh clean-up. Examples are presented showing the application of the algorithm.

1 Introduction

Finite element analysis (FEA) continues to push the limits of computing power in terms of the size of models being analyzed. The computation time required even by the most powerful machines can be hours or days on complex problems, with computational time increasing proportional to the cube of the number of nodes in the finite element mesh [1]. However, the accuracy of the finite element analysis is also proportional to the number of nodes in the mesh, increasing with finer resolution meshes. In many analysis situations, there may be specific areas in the mesh where accuracy, and thus mesh resolution, is more important than in other locations where lower resolution may be tolerated. For example, if a solution is shown to have high gradient in a particular location, the errors produced by a low density mesh may be significant and

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prevent proper engineering conclusions from the analysis process. Therefore, adapting the mesh to allow higher nodal density in the areas of interest can be critical to accurate analysis, while lower density in low gradient areas may allow for reductions in the solution time. If a uniform mesh is used, analysis time may be much greater than necessary while overall accuracy in the analysis may not improve significantly. The competing objectives of accuracy and computation time have prompted investigation into the field of mesh adaptation for the purpose of optimizing meshes.

Most research in mesh adaptation for computational mechanics centers on refinement algorithms that increase element density locally [2]. A complementary algorithm for decreasing local element density by removing elements (i.e., coarsening) could be a powerful companion tool to refinement algorithms, potentially allowing more flexible mesh adaptation. Unfortunately, coarsening is an area of research which has received limited attention. The algorithm presented in this paper, Automated Quadrilateral Coarsening by Ring Collapse (AQCRC), provides a fully automated conformal coarsening algorithm suitable for use with generalized unstructured quadrilateral finite element meshes.

Tools for mesh manipulation by both refinement and coarsening operations increase the ability to adapt a mesh. For example given a uniform mesh, the mesh density in an area of interest may be increased by established refinement techniques [3] and decreased away from the areas of interest using the coarsening technique described in this paper. An initial analysis on a base mesh may be used to indicate locations where high density meshes and low density meshes are appropriate based on gradients of the initial solution. Rather than remeshing the model, the base mesh may be modified using refinement and coarsening tools. This would allow increased resolution and accuracy in the results while maintaining a similar computation time for the entire model. Furthermore, a given model may require adaptation in different locations depending on different load cases, adaptation by both refinement and coarsening from a single base mesh may allow more efficient and robust generation of meshes appropriate for varied circumstances.

This paper describes the development and implementation of the AQCRC algorithm for use in quadrilateral finite element analysis. The background of the problem and discusses the limitations of previously developed coarsening and simplification algorithms is presented first, followed by a description of the implementation of the algorithm. A brief case study is then presented to show the merit of the algorithm and, finally, conclusions are drawn with recommendations of further research.

2 Background

Mesh adaptation is a field which has received extensive study among both computational mechanics and computer graphics researchers. Generally these two fields have not collaborated due to the many additional restrictions that apply to computational mechanics but are not necessary in computer graphics. The adaptation algorithms developed for computer graphics are, therefore, rarely suited to computational mechanics. One example of these additional restrictions in computational mechanics involves the requirement that a mesh must accurately represent the geometry of the model by insuring that the nodes representing a curve or surface of the model do not move off the geometry, whereas in graphics a sufficiently low level of detail might justify combining surfaces and/or curves. In the next section, we provide some background on efforts in mesh adaptation and give some additional motivation for our work.

2.1 Triangle Mesh Simplification and Coarsening

The use of triangular meshes in computer graphics and computational mechanics is common due to the relative simplicity of generating the meshes from these simplex elements. Triangle meshing algorithms are well-established and on-going efforts in the research community continue to advance the quality of these meshes.

One of the foremost algorithms of triangle mesh simplification was developed by Garland, et al. [4]. His approach is fast, reliable, and is also generally applicable to any polygon mesh. The algorithm assumes that the mesh is composed entirely of triangles or can be broken into a mesh entirely composed of triangles. It is designed to combine surfaces and curves that are indistinguishable when rendered at a low level of detail. Hoppe, et al. [5], demonstrate mesh adaptation respecting geometric curves and surfaces in order to preserve sharp corners and edges in the mesh representation.

A survey of triangle mesh coarsening algorithms is documented by Cignoni, et al. [6]. Cignoni et al. identify the following major simplification methodologies: coplanar facets merging, controlled vertex/edge/face decimation, retiling, energy function optimization, vertex clustering, wavelet based approaches, and simplification via intermediate hierarchical representation. Additional reviews that compare smaller sets of algorithms are also found in [7, 8].

While triangle meshes have widespread use, quadrilateral meshes are sometimes preferred in computational analysis due to beneficial mathematical properties of the quadrilateral element that result in reduced solution error with fewer elements than triangle meshes [1]. Unfortunately, despite the wide availability of triangle mesh adaptation algorithms, most of the algorithms developed for triangle meshes cannot be adapted for use on quadrilateral meshes.

2.2 Quadrilateral Mesh Simplification and Coarsening

Various efforts have been made to develop all-quadrilateral coarsening algorithms; unfortunately, all of these algorithms have significant restrictions which prevent use with unstructured meshes. Takeuchi, et al. [9], modified the approach developed by Garland, et al. [4], to simplify quadrilateral meshes; however, the process is designed for full-model simplification and may produce degenerate elements (i.e. quadrilaterals which are inverted or concave). Cheng, et al., developed a method of coarsening a structured, all-quadrilateral mesh specifically for use on auto-body parts [10]; however, this method has not been adapted for use in unstructured meshes. Kwak, et al., performs simplification using remeshing algorithms [11]; however, this global approach can be slow when only local adaptation is needed. Choi describes an algorithm which can be used to undo previous refinement on both quadrilateral and hexahedral meshes [12]; however, the reliance on knowledge of previous refinement restricts the algorithm from being used on a base mesh that has not been refined. Nikishkov developed a quadtree method for mesh adaptation that allows both refinement and coarsening [13]; however, his method requires the use of special elements or else it produces non-conforming elements. The left panel of **Fig. 1** shows an initial uniform mesh with the region to be coarsened highlighted in grey. The center panel shows the mesh once Nikishikov's coarsening method has been applied. The nodes marked A, B, C, and D are locations where the mesh is not conforming. We note that conforming quadrilateral meshes, where the elements are simply connected and there are no gaps, are required by most FEA solvers.

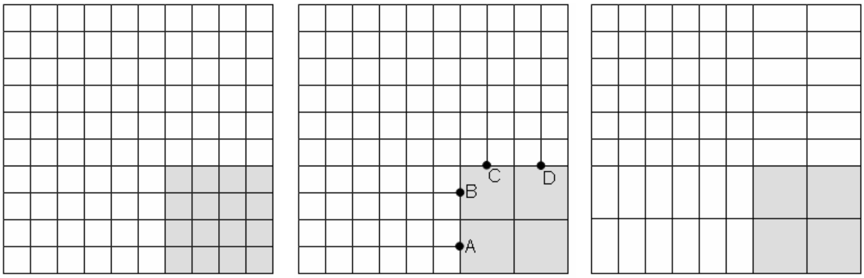

Fig. 1. Quadtree and chord removal coarsening

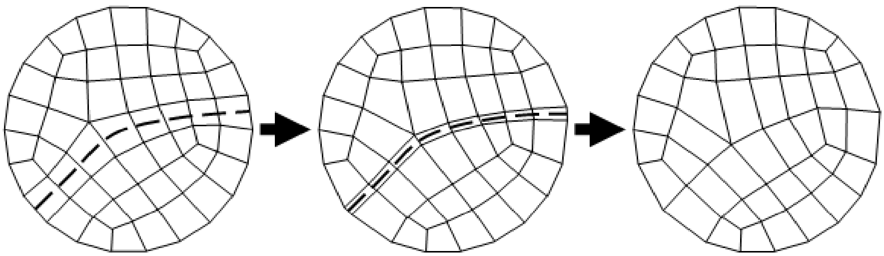

Fig. 2. Removing a chord from a mesh

The basis of the AQCRC algorithm derives from the dual chord representation of a quadrilateral/hexahedral mesh [14,21]. A dual chord is a set of quadrilaterals connected through pairs of opposite edges extending through the mesh or connecting back on the original starting edge. In **Fig. 2** a dashed line is shown highlighting one chord of the mesh.

Borden, et al. [15], recognized that it is possible to remove an entire chord from a quadrilateral mesh, maintaining conformal connectivity, by simply collapsing the defining edges of the chord as shown in **Fig. 2.** The removal of a chord reduces the number of quadrilaterals in the mesh and coarsens the quadrilaterals adjacent to the chord. The right panel of **Fig. 1.** shows this coarsening applied to the selected coarsening region. Unfortunately, while the local region, highlighted in gray, is significantly coarsened, the effect of the coarsening may extend well beyond the boundaries of the coarsening region. The research of Borden, et al., was continued in the paper by Benzley, et al. [16], where initial steps were made to localize the coarsening region.

Staten, et al. [17], speculated that if a circular chord (i.e. a chord which forms a closed loop) could be created in a specific region, then the removal of that chord would produce localized coarsening. Staten, et al., showed that if the portions of chords bounding a region can be established, then simple chord operations (i.e. alterations to the mesh which change the connectivity of chords) can be performed at the intersections of these bounding partial chords to combine them into a single continuous chord surrounding the region to be coarsened. The operations used to combine the partial chords into a single circular chord included the edge swap, face close and doublet insertion operations that are described in the following paragraph. The faces within the region to be coarsened in these partial chords form a ring of quadrilaterals. A ring of quadrilaterals within the coarsening region will be known as a 'coarsening ring' throughout the remainder of this paper.

Fig. 3 shows an example of the coarsening process developed by Staten, et al. [17]. The left panel shows the initial mesh. The dashed lines mark chords which bound the region to be coarsened. The quadrilaterals highlighted in grey are the bounding partial chords and define a coarsening ring. In the second panel, the quadrilaterals where two bounding partial chords intersect have been modified with doublet insertion, face close, and edges swap operations. The top left intersection is modified using a doublet insertion operation. The doublet insertion operation causes the quadrilateral at the intersection of two bounding chords to be divided into two degenerate quadrilaterals by inserting two edges and a node between opposite nodes on a single quadrilateral as shown in detail in the bottom left corner of **Fig. 4**. In both figures, the doublet node is circled for clarity. A doublet node is an internal node in a quadrilateral mesh connected to only two edges. The bottom two chord intersections in **Fig. 3** are modified by a face close operation. The face close operation results in the quadrilateral at the intersection of the bounding chords to be deleted by merging two nodes opposite each other as shown in detail in the top right corner of **Fig. 4**. The top right chord intersection in **Fig. 3** is modified by an edge swap operation. The edge swap operation results in an edge between the quadrilateral at the intersection of the bounding chords and one of the adjacent quadrilaterals within one of the bounding chords to change node connectivity as shown in detail in the bottom right corner of **Fig. 4**. The combination of these operations result in the mesh shown in **Fig. 3**, which contains a single circular chord that now bounds the coarsening region as shown in the center panel of the same figure. This chord is removed by an extraction operation (see **Fig. 2.**), resulting in the mesh shown in the right panel of **Fig. 3**.

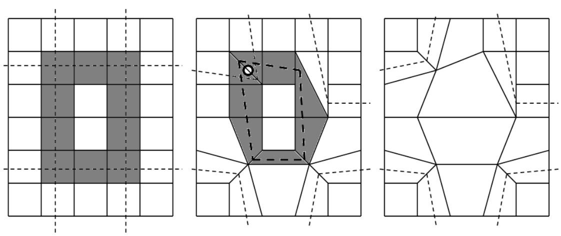

Fig. 3. Chord operations and removal

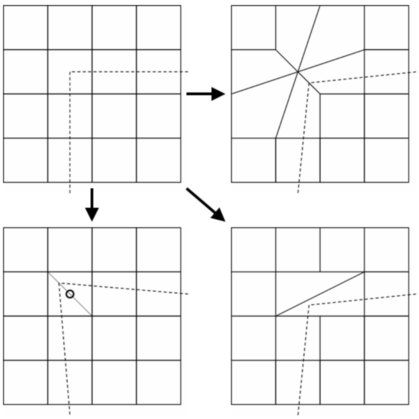

Fig. 4. Chord operations: top right, Face Close; bottom left, Doublet Insertion; bottom right, Edge Swap

3 Automated Quadrilateral Coarsening by Ring Collapse

The algorithm developed by Staten, et al., is a coarsening algorithm which can locally coarsen an unstructured, conforming, all-quadrilateral mesh [17]. The AQCRC algorithm, presented in this Section, generalizes, further develops and automates the work of Staten, et al. The AQCRC algorithm advances the basic methodology by extracting multiple coarsening rings simultaneously, introducing logic to maximize element quality, and eliminating the step of combining the bounding partial chords into a single circular chord.

One of the key developments of the AQCRC algorithm is the use of coarsening rings, i.e. a closed ring of quadrilaterals contained within the coarsening region, rather than circular chords. The AQCRC algorithm does not take the step of modifying the bounding partial chords with chord operations to create a single chord (see **Fig. 3.**). Instead the coarsening ring is removed directly.

The AQCRC algorithm iterates over five steps until sufficient coarsening has been accomplished. These five steps are outlined below and will be examined in detail in the following sections.

1. A contiguous coarsening region and the final mesh coarseness are specified.
2. One or more coarsening rings are selected within the coarsening region containing a number of quads less than or equal to a goal number of quads.
3. The bounding partial chord intersections may be altered with chord operations to increase the final quality of the mesh or to prevent merging nodes illegally.
4. The identified coarsening rings are collapsed from the mesh and the mesh reconnected in a manner that retains its conformal properties.
5. The mesh is checked, cleaned up, and smoothed to ensure that elements have acceptable quality.

3.1 Defining the Coarsening Region and Removal Parameter

The first step in the AQCRC algorithm is to specify a coarsening region. The left panel of **Fig. 5** shows an example of a coarsening region highlighted in dark gray. An additional layer of elements marked with light gray surrounds the coarsening region and is used as a transition layer. Elements within the coarsening region may be moved and/or deleted during the coarsening process; the elements in the transition region may be moved slightly, but not deleted. If the coarsening region includes a boundary of the initial mesh (i.e. a curve which only bounds one meshed surface), the elements along that boundary are also considered transition elements.

In addition to specifying the coarsening region, the number of quadrilaterals to be removed must be defined. A target size or coarsening factor may be given to establish the goal number of quadrilaterals which are to be removed from the mesh. A coarsening factor corresponds to the multiplicative increase in average area that should occur within the coarsening region during the execution of the AQCRC algorithm. A target size corresponds to the average length of the edges in the coarsening region after the execution of the AQCRC algorithm.

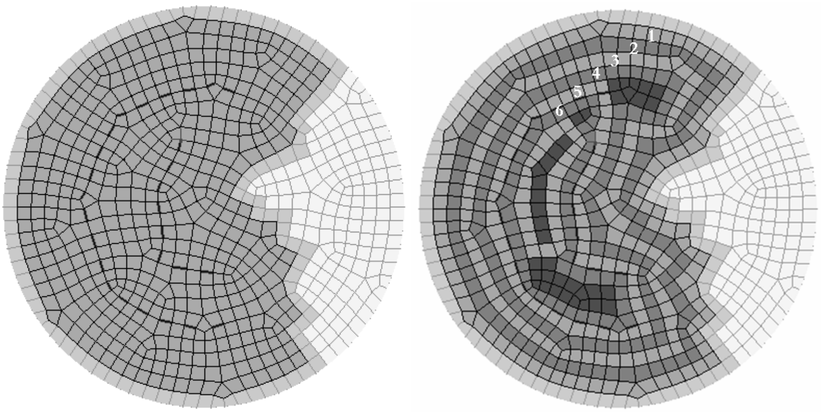

Fig. 5. Coarsening region selection and coarsening rings

Equation 1 shows how a coarsening factor is used to determine the number of quadrilaterals that should be removed.

$$N_{e-r} = E_t - \frac{E_t}{F} \tag{1}$$

where N_{e-r} = the number of elements to be removed

E_t = the number of elements in the coarsening region

F = coarsening factor

Equation 2 converts the target size into an equivalent coarsening factor which is then converted into the number of elements to be removed by Equation 1.

$$F = \frac{l_f^2}{l_0^2} \tag{2}$$

where F = coarsening factor
l_f = the final average edge length specified by the user
l_0 = the initial average edge length in the coarsening region

Once the number of quadrilaterals to be removed is determined, a 10% tolerance factor (t_f) is also calculated such that the number of quadrilaterals actually removed is within ± t_f of the calculated goal. This tolerance factor is limited to being a minimum of 3 quadrilaterals or a maximum of 50 quadrilaterals. The values of 10%, 3 and 50 are heuristic values shown to give reasonable results during our experimentation.

3.2 Selecting Coarsening Rings

Once a coarsening region has been defined, a set of concentric coarsening rings is developed for removal. Each of the coarsening rings is analyzed to determine which coarsening ring removals will preserve a high quality mesh and the goal number of elements to be removed.

Ring Identification
A coarsening ring is identified by locating an element in the coarsening region which is adjacent to or shares a node with an element marked as a boundary element. Because the coarsening region is always bounded, the set of elements adjacent to the boundary forms a coarsening ring. The elements in the new coarsening ring are set aside forming a new, but smaller, coarsening region. This process is iterated, until all coarsening rings in the coarsening region are identified. During the process of identifying coarsening rings there are a few cases where the ring identified could not be collapsed or there are not enough elements to form a closed loop. These invalid cases are handled by marking some of the elements as boundary elements. Eventually all of the elements in the coarsening region will be marked as boundary elements, which effectively ends the ring identification portion of the algorithm. In most cases, the coarsening region is large enough that several rings are created concentrically. The right panel of **Fig. 5** shows the rings developed within the coarsening region. The alternating numbered regions of darker and lighter shaded grey elements show the set of rings. The dark regions not numbered are locations of elements which are not included as rings because they were part of an invalid ring case. Further details on handling invalid cases can be found in [18].

Ring Selection
Once the set of coarsening rings has been created, a subset is chosen for removal. To facilitate choosing an optimum set of rings for removal, each node connected to a ring is assigned to a node group. A node group is the set of nodes that will be merged into a single node when the coarsening ring is collapsed. The node groups are used to facilitate choosing the best set of rings to collapse by providing a rough estimate of the mesh connectivity and element shapes that will be created upon collapse. Each node group will be assigned a projected location. Generally, the projected location is the

centroid of all the nodes in the group. This location may be modified so that the mesh continues to adhere to the geometry of the model.

The projected location of the node groups is calculated to enable quality metric calculations for each of the coarsening rings. The rings are then ordered based on prospective quality. Coarsening rings are removed based on the ordered list until the coarsening goal is satisfied (within the tolerance factor).

3.3 Improving Quality

In some cases, modifying the connectivity of a ring can improve the quality of the final mesh. As each ring is chosen for collapse, it is examined more closely to see if there is room for quality improvements. Quality is a major consideration in coarsening because ring removal generally reduces mesh quality. Quality improvement operations typically prevent the creation of high-valence nodes, (e.g., nodes with more than 5 edges), as well as to prevent merging nodes from different curves (see [17, 18] for details).

3.4 Collapsing Coarsening Rings

Once the coarsening rings have been selected and the various quality improvement operations have been applied, the mesh is ready to be coarsened. Each of the selected coarsening rings is collapsed in succession using the following procedure. The nodes in each node group are moved to the projected location and the quadrilaterals that are part of the ring are deleted. As the quadrilaterals are deleted, any edge that is no longer associated with a quadrilateral (i.e. the quadrilaterals on either side of it have been removed) is deleted and the nodes on either end of the deleted edge are merged together. At corners, a simple collapse as shown in **Fig. 6** creates a conformal mesh (compare to the operation shown in **Fig. 3.**). In the left and center panels of **Fig. 6**, the node groups are circled with darkened edges connecting each group. The dashed line indicates the coarsening ring of quadrilaterals being collapsed. In the right panel the circled nodes are the locations of the merged nodes in the final mesh.

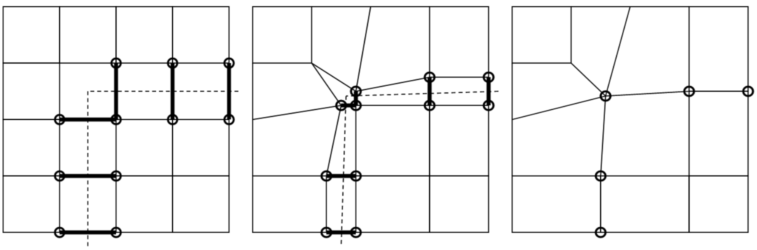

Fig. 6. Collapsing node groups

3.5 Mesh Clean-Up

Despite efforts to minimize high-valence nodes during quality improvement, a few cases remain where high-valence nodes are formed. Furthermore, the collapse of two rings separated by a single layer of elements may reduce element quality due to multiple node projections for quadrilaterals in the non-removed coarsening ring. Additionally, collapsing quadrilaterals along geometric curves may result in elements with low quality which cannot be smoothed. To improve mesh quality, clean-up operations like those described in [18, 20], which change mesh connectivity to create a more structured mesh, and smoothing operations, like those described in [23], which improve the mesh quality by simply moving node coordinates, are applied.

3.6 Coarsening Iterations

At the end of each coarsening cycle the net number of quadrilaterals removed is determined. This net number includes any quadrilaterals added or removed by the clean-up procedures. If too few quadrilaterals have been removed, the algorithm is executed again. If a sufficient number of quadrilaterals have been removed, or if no quadrilaterals have been removed, the new, coarsened mesh is integrated into the original mesh. The algorithm provides a message if insufficient coarsening has taken place to reach the goal number of elements to be removed. **Fig. 7** shows the example given in **Fig. 5** after it has gone through several coarsening iterations. The region exterior to the coarsening region is identical in both meshes.

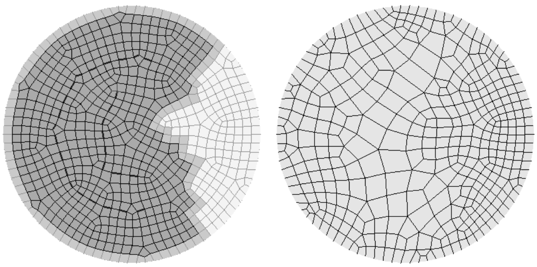

Fig. 7. Original mesh compared to coarsened region

4 Results and Example

One example of mesh coarsening using the AQCRC algorithm is given in this section. The resulting mesh and quality of the initial mesh and of a coarsened mesh will be compared. The quality metric used is the scaled Jacobian metric [19] which ranges from-1.0 to 1.0, where a value of 1.0 represents a perfect square while anything below 0.0 is an inverted (non-convex) element (0.0 typically being a triangle-shaped element). A scaled Jacobian value greater than 0.2 is generally considered acceptable for analysis accuracy. A scaled Jacobian value below 0.2 are considered marginal [21].

4.1 Lever Mesh Example

Fig. 8. shows a lever model that has been meshed with shell elements. An initial mesh and a mesh that has had most of the inside area coarsened to a factor of four are

shown. Previous analysis showed that high stress gradients occurred around the outside boundaries. In this example the mesh has had an analysis run on it using Calculix, an open source finite element analysis package. Both analyses had nearly identical results; the error introduced by the coarsening is less than 0.2% while the calculation speed increased 347%. Table 1 shows the element count of the original and coarsened meshes, the minimum scaled jacobian of the elements in the mesh, the analysis time and peak Von Mises Stress of the model. The accuracy of the analysis has not been compromised, but the speed of analysis has increased by more than three.

Table 1. Lever mesh results

Mesh	Element Count	Min. Scaled Jacobian	Analysis Time	Peak Von Mises Stress
Initial	11113	0.71	3m 11s	5.34 E+4 psi
Coarsened (factor 4)	3261	0.39	55s	5.33 E+4 psi

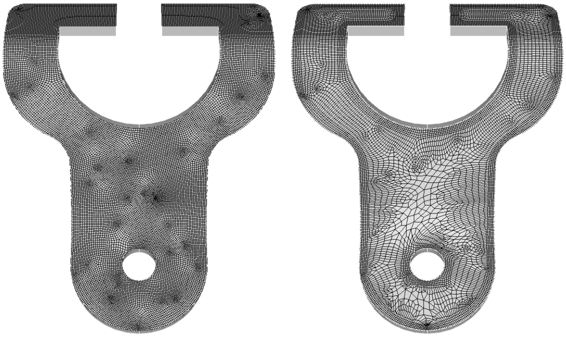

Fig. 8. Lever shell mesh

5 Conclusions and Recommendations

The level of coarsening to be achieved is limited primarily by element quality considerations. Generally speaking, the more the mesh is coarsened, the greater the reduction in mesh quality. This can largely be attributed to the quality of the transition elements between the fine and coarse regions of the mesh. At least one layer of elements must take on a trapezoidal shape to enable the sizing transition. As the aspect ratio increases, the quality of the mesh is reduced. This should not deter from the use of the coarsened mesh so long as the quality remains within acceptable ranges.

The availability of a fast, robust coarsening algorithm for unstructured, all-quadrilateral, conformal meshes expands the tools available for computational modeling. The use of coarsening coupled with refinement provides and effective means to adapt a mesh more effectively, increasing model accuracy while reducing computation times.

This paper presented the Automated Quadrilateral Coarsening by Ring Collapse (AQCRC) algorithm. The algorithm conformally coarsens localized regions of quadrilateral meshes by creating, modifying and removing coarsening rings. The automation procedure optimizes final element quality while attempting to heavily coarsen the defined region. Application of the AQCRC algorithm demonstrates that it is capable of removing enough nodes from the mesh to increase speeds by many times while maintaining a mesh quality sufficient for accurate analysis. The potential speed benefits of coarsening algorithms could significantly reduce the time and cost of computational analysis of large scale problems.

5.1 Further Research and Development

Further research and development of this algorithm continues. Clean-up operations are already a focus of significant effort to address problems coarsening around complex surfaces. An alternative method of coarsening, partial chord removal, is also being explored.

One of the methods of coarsening discussed in the Section 2 was undoing refinement. The AQCRC algorithm could be developed into a reversible coarsening procedure, allowing refinement of regions previously coarsened. This could be accomplished by storing the changes made to the mesh in the remaining mesh entities. As some nodes or edges may be merged and then merged again, a set of changes could be pushed onto a stack so that the reverse operations could be performed one at a time, restoring the mesh to its original state.

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