Hessian-free metric-based mesh adaptation via geometry of interpolation error

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1 Introduction

Generation of meshes adapted to a given function u requires a specially designed metric. For metric derived from the Hessian of u, optimal error estimates for the interpolation error on simplicial meshes have been proved in [2, 5, 8, 10, 11]. The Hessian-based metric has been successfully applied to adaptive solution of PDEs [4, 7, 9]. However, theoretical estimates have required to make an additional assumption that the discrete Hessian approximates the continuous one in the maximum norm. Despite the fact that this assumption is frequently violated in many Hessian recovery methods, the generated adaptive meshes still result in optimal error reduction.

In this article we continue the rigorous analysis [1, 3] of an alternative way for generating a space tensor metric using the error estimates prescribed to *mesh edges*. The new methodology produces meshes resulting in the optimal reduction of the P_1-interpolation error or its gradient. We define a tensor metric $\mathfrak{M}$ such that the volume and the perimeter of a simplex measured in this metric control the norm of error or its gradient. The equidistribution principle, which can be traced back to D'Azevedo [6], suggests to balance $\mathfrak{M}$-volumes and $\mathfrak{M}$-perimeters. This leads to meshes that are quasi-uniform in the metric $\mathfrak{M}$.

The paper outline is as follows. In Section 2, we derive appropriate metrics from analysis of the interpolation errors. In Section 3, we present the algorithm for generating adaptive meshes and its application to a model problem.

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2 Metric derivation from local error analysis

Let $\Omega \subset \Re^d$ be a bounded polyhedral domain and Ω_h be a conformal simplicial mesh with N_h simplexes. Let $\mathfrak{M}$ be a piecewise constant tensor metric on Ω_h.

The volume of simplex Δ and the total length of its edges in this metric are denoted by $|\Delta|_{\mathfrak{M}}$ and $|\partial\Delta|_{\mathfrak{M}}$, respectively [2].

Let $\mathcal{I}_1 u$ be the piecewise linear interpolant of u, and $\mathcal{I}_{1,\Delta} u$ be its restriction to Δ. Our goal is to generate meshes that minimize the L_p-norm, $p \in (0, \infty]$, of the interpolation error $e = u - \mathcal{I}_1 u$ or its gradient ∇e.

Let us consider a particular d-simplex Δ with vertices $\mathbf{v}_i$, $i = 1, \dots, d+1$, edge vectors $\mathbf{e}_k = \mathbf{v}_i - \mathbf{v}_j$, $1 \le i < j \le d+1$, and mid-edge points $\mathbf{c}_k$, $k = 1, \dots, n_d$, where $n_d = d(d+1)/2$. Let λ_i, $i = 1, \dots, d+1$, be the linear functions on Δ such that $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ where δ_{ij} is the Kronecker symbol. For every edge $\mathbf{e}_k$, we define the quadratic bubble function $b_k = \lambda_i \lambda_j$.

Step 1. We begin with the derivation of a tensor metric from edge data.

Lemma 1 (Metric existence [1]). *Let α_k, $k = 1, \dots, n_d$, be values prescribed to edges of a d-simplex Δ such that $\alpha_k \ge 0$ and $\sum_{k=1}^{n_d} \alpha_k > 0$. Then, there exists a constant tensor metric $\mathfrak{M}_\Delta$ such that*

$$\left(\frac{d!}{(d+1)(d+2)} \right)^{1/d} |\Delta|_{\mathfrak{M}_\Delta}^{2/d} \le \sum_{k=1}^{n_d} \alpha_k \le |\partial\Delta|_{\mathfrak{M}_\Delta}^2. \tag{1}$$

Step 2. Let u_2 be a continuous piecewise quadratic function and $e_2 = u_2 - \mathcal{I}_{1,\Delta} u_2$ be the linear interpolation error. We have

$$e_2 = 4 \sum_{k=1}^{n_d} (u_2(\mathbf{c}_k) - \mathcal{I}_{1,\Delta} u_2(\mathbf{c}_k))\, b_k = \sum_{k=1}^{n_d} \gamma_k\, b_k,$$

where $\gamma_k = 4(u_2(\mathbf{c}_k) - \mathcal{I}_1 u_2(\mathbf{c}_k))$. The L_2-norm of the error e_2 is given by

$$\|e_2\|_{L_2(\Delta)}^2 = |\Delta|\, (\mathbb{B}\, \boldsymbol{\gamma},\, \boldsymbol{\gamma}), \tag{2}$$

where $\boldsymbol{\gamma}$ is a vector with n_d components γ_k and $\mathbb{B}$ is the $n_d \times n_d$ symmetric positive definite Gramm matrix with positive entries

$$\mathbb{B}_{k,l} = \frac{1}{|\Delta|} \int_\Delta b_k b_l \,\mathrm{d}V.$$

Note that $\mathbb{B}$ is spectrally equivalent to the identity matrix. Thus,

$$c_1\, |\Delta| \left(\sum_{k=1}^{n_d} |\gamma_k| \right)^2 \le \|e_2\|_{L_2(\Delta)}^2 \le c_2\, |\Delta| \left(\sum_{k=1}^{n_d} |\gamma_k| \right)^2, \tag{3}$$

where the constants $c_2 \ge c_1 > 0$ depend only on the space dimension d.

Analysis of the L_2-norm of ∇e_2 in [1] uses the Cholesky decomposition of $\widetilde{\mathbb{B}} = \mathbb{L}\mathbb{L}^T$, the Gramm matrix for vector-functions ∇b_k, to get:

$$\|\nabla e_2\|_{L_2(\Delta)}^2 = |\Delta| \sum_{k=1}^{n_d} \beta_k^2, \tag{4}$$

where $\boldsymbol{\beta} = \mathbb{L}^T \boldsymbol{\gamma}$. Thus, the L_2-norms of e_2 and ∇e_2 are controlled by a sum of non-negative numbers associated with the edges of simplex Δ times $|\Delta|$.

Using Lemma 1 we build the metric $\mathfrak{M}_\Delta$ for e_2 by setting $\alpha_k = |\gamma_k|$. Similarly, to build the metric $\widetilde{\mathfrak{M}}_\Delta$ for ∇e_2, we set $\alpha_k = \beta_k^2$. In the next step we convert these metrics to optimal metrics for the L_p-norm.

Step 3. The extension of error estimates to general L_p-norms follows the path described in [1]. With a slight modification of the argument used there, we may show that the *optimal metrics* for the L_p-norm of e_2 and ∇e_2 are:

$$\mathfrak{M}_{\Delta,p} = (\det(\mathfrak{M}_\Delta))^{-1/(d+2p)}\,\mathfrak{M}_\Delta \quad \text{and} \quad \widetilde{\mathfrak{M}}_{\Delta,p} = (\det(\widetilde{\mathfrak{M}}_\Delta))^{-1/(d+p)}\widetilde{\mathfrak{M}}_\Delta.$$

For simplicity, we confine ourselves to the case $p = \infty$. In this case, the metrics generated by Lemma 1 are optimal, i.e. $\mathfrak{M}_{\Delta,\infty} = \mathfrak{M}_\Delta$ and $\widetilde{\mathfrak{M}}_{\Delta,\infty} = \widetilde{\mathfrak{M}}_\Delta$.

Step 4. For a given continuous function u, we define a computable error e_2 which will be used to estimate the true error e_Δ:

$$e_2 = \mathcal{I}_{2,\Delta} u - \mathcal{I}_{1,\Delta} u \qquad \text{and} \qquad e_\Delta = u - \mathcal{I}_{1,\Delta} u,$$

where $\mathcal{I}_{2,\Delta} u$ be the piecewise quadratic Lagrange interpolant of u on Δ.

Let $\mathcal{F}$ be the space of symmetric $d \times d$ matrices and $|\mathbb{H}|$ be the spectral module of $\mathbb{H} \in \mathcal{F}$. We introduce the following notations:

$$|||\mathbf{e}_k|||^2_{|\mathbb{H}|} = \max_{\mathbf{x}\in\Delta} (|\mathbb{H}(\mathbf{x})|\mathbf{e}_k, \mathbf{e}_k) \qquad \text{and} \qquad |||\partial\Delta|||^2_{|\mathbb{H}|} = \sum_{k=1}^{n_d} |||\mathbf{e}_k|||^2_{|\mathbb{H}|}.$$

Lemma 2 (L_2 error). *Let $u \in C^2(\bar{\Delta})$. Then*

$$\frac{d+1}{2d}\|e_2\|_{L_\infty} \le \|e_\Delta\|_{L_\infty} \le \|e_2\|_{L_\infty} + \frac{1}{4}\inf_{\mathbb{F}\in\mathcal{F}} |||\partial\Delta|||^2_{|\mathbb{H}-\mathbb{F}|}.$$

Lemma 3 (Gradient error [1]). *Let $u \in C^2(\bar{\Delta})$. Then, there exist positive constants c_s, C_s, and $c_4 = c_4(d)$ such that*

$$c_s\|\nabla e_2\|_{L_\infty} - \operatorname{osc}(\mathbb{H},\Delta) \le \|\nabla e_\Delta\|_{L_\infty} \le C_s\|\nabla e_2\|_{L_\infty} + \operatorname{osc}(\mathbb{H},\Delta), \qquad (5)$$

where the oscillation term is

$$\operatorname{osc}(\mathbb{H},\Delta) = c_4\frac{|\partial\Delta|^{d-1}}{|\Delta|}\inf_{\mathbb{F}\in\mathcal{F}} |||\partial\Delta|||^2_{|\mathbb{H}-\mathbb{F}|}$$

The oscillation terms are conventional in contemporary error analysis. Their value depend on the simplex and particular features of the function. For instance, if $u \in C^2(\bar{\Delta})$, and Δ is shape regular, then $\operatorname{osc}(\mathbb{H},\Delta) \sim |\partial\Delta|^2$.

3 Metric-based mesh adaptation

We use the Algorithm 1 to build an adaptive mesh minimizing the L_p-norm of error or its gradient. The algorithm is more robust for continuous tensor

Algorithm 1 Adaptive mesh generation

```
Generate an initial mesh Ω_h and compute the metric 𝔐_p.
loop
    Generate a 𝔐_p-quasi-uniform mesh Ω_h.
    Recompute the metric 𝔐_p.
    If Ω_h is 𝔐_p-quasi-uniform, then exit the loop .
end loop
```

metrics that provide faster convergence and result in smoother meshes. We suggest two methods for recovering of a continuous nodal-based metric from the discontinuous piecewise-constant metric $\mathfrak{M}_p$.

For every node $\mathbf{a}_i$ in Ω_h, we define the superelement σ_i as the union of all d-simplices sharing $\mathbf{a}_i$. The first method is based on simple shifting: to every node $\mathbf{a}_i$, we assign the metric with the largest determinant from all metrics available in superelement σ_i. The second method is generalization of the ZZ-recovery method [12]. On every superelement σ_i, we search for a polynomial u_3 containing only cubic and quadratic terms. Let $\mathbb{H}_3$ be the Hessian of u_3. The free parameters are chosen to minimize the functional

$$\sum_{1\le j\le k\le d} \sum_{\Delta\in\sigma_i} \left(\mathbb{H}_{3,jk}(\mathbf{b}_\Delta) - (\mathfrak{M}_\Delta)_{jk}\right)^2,$$

where $\mathbf{b}_\Delta$ is the barycenter of simplex Δ. We set $\mathfrak{M}(\mathbf{a}_i) = \mathbb{H}_3(\mathbf{a}_i)$. To generate a $\mathfrak{M}$-quasi-uniform mesh, we use local mesh modifications described in [2, 4, 10] and implemented in package `Ani2D` (sourceforge.net/projects/ani2D).

In $\Omega = [0, 1]^2$ we consider the analytical function proposed in [6]:

$$u(x, y) = \frac{(x-0.5)^2 - (\sqrt{10}y + 0.2)^2}{((x-0.5)^2 + (\sqrt{10}y+0.2)^2)^2}.$$

The function has an anisotropic singularity at point $(0.5, -0.2/\sqrt{10})$ located outside the computational domain but close to its boundary. Table 1 shows that the L_∞-norm of the interpolation error is proportional to N_h^{-1}, while the L_∞-norm of its gradient is proportional to $N_h^{-0.5}$. Note that the meshes minimizing the interpolation error and its gradient are different (see Fig. 1).

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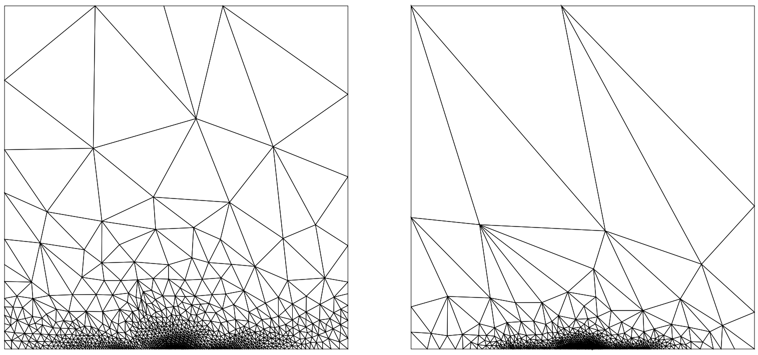

Fig. 1. The adaptive meshes with 2000 triangles minimizing the maximum norm of the interpolation error (left) and its gradient (right).

	Method of shifts		ZZ-type method	
N_h	$\|e\|_{L_\infty(\Omega)}$	$\|\nabla e\|_{L_\infty(\Omega)}$	$\|e\|_{L_\infty(\Omega)}$	$\|\nabla e\|_{L_\infty(\Omega)}$
1000	1.55e-1	6.55e+1	3.74e-1	9.49e+1
4000	4.64e-2	3.16e+1	6.83e-2	4.91e+1
16000	1.14e-2	1.71e+1	2.00e-2	2.79e+1
64000	3.33e-3	8.39e+0	8.14e-3	1.34e+1
rate	0.93	0.49	0.92	0.47

Table 1. Convergence of the interpolation error and its gradient.

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