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Automatic Extraction of Quadrilateral Patches from Triangulated Surfaces Using Morse Theory

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Summary. A method for decompose the triangulated surface into quadrilateral patches using Morse theory and Spectral mesh analysis is proposed. The quadrilateral regions extracted are then regularized by means of geodesic curves and fitted using a B-splines creating a new grid on which NURBS surfaces can be fitted.

1 Introduction

One of the most important phases in the reconstruction of 3–D model from triangular meshes is the the process of fitting high-level surface primitives such as NURBS (Non-Uniform Rational B-Splines) from them. Triangular meshes often exhibit several deficiencies. They are often too large or too high-resolution and typically contain elements with inadequate shapes. On the other hand, NURBS have become the standard in modern CAD/CAM systems, because of their ease of use and and their ability to deal at high-level with local surface modifications. Although NURBS have the capacity to represent arbitrarily curved surfaces, they still present problems when one wants to model fine details. This is due to how the NURBS surface are defined where parameters such as: knots, weights, and control points must be controlled by the CAD user to achieve a certain shape. In addition, NURBS surfaces also require to be placed on a networks of curves that usually have a quadrilateral topology.

The majority of the work reported in the literature on re-meshing methods, is focused on the problem of producing well formed triangular meshes (ideally Delaunay). However, the ability to produce quadrilateral meshes is of great importance as it is a key requirement to fit NURBS surface on a large 3-D mesh. Quadrilateral topology is the preferred primitives for modelling many objects and in many application domains. Many formulations of surface subdivision such as SPLINES and NURBS, require complex quadrilateral bases. Recently, methods to automatically quadrilateralize complex triangulated mesh have been developed such as the one proposed by Dong et al. [5]. These methods are quite complex, hard to implement, and have many heuristic components.

A method for decompose the triangulated surface into quadrilateral patches using Morse theory and Spectral mesh analysis is proposed. The quadrilateral regions obtained from this analysis is then regularized by means of computing the geodesic curves between each corner of the quadrilateral regions and then a B-splines curves are fitted to the geodesic curves on which NURBS surfaces are fitted. Such NURBS surfaces are optimized by means of evolutive strategies to guaranty the best fit as well as C^1 continuity between the patches.

This paper is organized as following: In Section 2, an introduction to Morse theory is presented. In Section 4, the proposed method for the adjustment of surfaces by means of optimized NURBS patches is presented. In Section 7.1 is presented a comparison between Branch's method and Eck and Hoppe's method. In Section 7, the results with the proposed model are discussed, and finally in Section 8, conclusions are presented.

2 Morse Theory

In a general way, spectral mesh analysis tries to infer topological features of the object through mathematical functions. This produces a spectrum which becomes a set of eigenvectors and eigenvalues of a matrix which has been inferred from the triangular mesh. The spectral analysis is supported by the Morse Theory. This theory chooses some representative points from the vertexes of the triangular mesh, critical Morse points.

Given a real function on a surface, Morse theory connects the differential geometry of a surface with its algebraical topology. This theory describes the connectivity of the surface from the configuration of the points where the gradient function decays. Such points are called critical points (these are: minimum, maximum and saddle points). The Morse theory has been used by the graphics and computer visualization community to analyze different real functions. For example, in terrain data analysis, Morse theory is used to identify topological features, while controlling the plane's simplification and organizing the features in a multi-resolution hierarchy [1], [4].

Let S be a smooth, compact 2-manifold without a boundary and let $h : S \rightarrow \mathbb{R}$ be a smooth map. The differential of h at the point a is a linear map $dh_a : TS_a \rightarrow TS_{h(a)}$, mapping the tangent space of S at a to that of $\mathbb{R}$ at $h(a)$ (The tangent space of $\mathbb{R}$ at a point is simply $\mathbb{R}$ again, with the origin shifted to that point). In a formal way: let $a \in S \subset \mathbb{R}^n$ a point in a continuous neighborhood parametrized by (u, v) [5]: a point $a \in S$ is called *critical* $h(a)$, if $h_u(a) = h_v(a) = 0$ (its calculated two partial derivatives, and are called critical when both are zero), otherwise, it is a regular point. The critical point a is degenerate if $h_{uu}(a)h_{vv}(a) - 2h_{uv}(a) = 0$, otherwise, it is a Morse point. If al critical points satisfies the Morse conditions, then the function h is a Morse function.

At a critical point a, we compute the local coordinates of the Hessian of h:

$$H(a) = \begin{bmatrix} \frac{\partial^2 h}{\partial x^2}(a) & \frac{\partial^2 h}{\partial y \partial x}(a) \\ \frac{\partial^2 h}{\partial x \partial y}(a) & \frac{\partial^2 h}{\partial y^2}(a) \end{bmatrix}. \quad (1)$$

The Hessian is a symmetric bilinear form on the tangent space TS_a of S at a. The matrix above expresses this function in terms of the basis $(\frac{\partial}{\partial x}(a), \frac{\partial}{\partial y}(a))$ for TS_a. A critical point a is called *non-degenerate* if the Hessian is non-singular at a; i.e., $detH(a) \neq 0$, a property that is independent from the coordinate system. The Morse Lemma [7], states that near a non-degenerate critical point a, it is possible to choose local coordinates so that h takes the form:

$$h(x, y) = h(a) \pm x^2 \pm y^2. \quad (2)$$

The number of minuses is called *index i(a)* of h at a, and equals the number of negative eigenvalues of $H(a)$ or, equivalently, the index of the functional $H(a)$. The existence of these local coordinates implies that non-degenerate critical points are isolated.

Let $\lambda_1 \leq \lambda_2$ be the two eigenvalues of the Hessian of h, with corresponding eigenvectors. The index of a critical Morse point is the number of negative eigenvalues of its Hessian. Therefore, this can be classified as: minimum, (index 0, h increases in every direction), saddle point (index 1, h changes in decrements and increments four times around a point), and maximum (index 2, h decreases in every direction). The function h is called Morse function if its critical points are not degenerated.

3 Morse Theory for Triangular Meshes

The application of the Morse theory for triangular meshes implies to discretize Morse analysis. The Laplacian equation is used to find a Morse function which describes the topology represented on the triangular mesh. In this sense, additional points of the feature of the surface might exist, which produce a basis domain which adequately represents the geometry of the topology itself and the original mesh. The mesh can also be grouped into improved patches. In this work, Morse theory is applied by representing the saddle points and its borders by a Morse function which can then be used to determined a number of critical points.

This approximation function is based on a discrete version of the Laplacian, to find the harmonic functions. In many ways, Morse theory relates the topology of a surface S with its differential structure specified by the critical points of a Morse function $h : S \rightarrow \mathbb{R}$ [11] and is related to the mesh spectral analysis.

The spectral analysis of the mesh is performed by initially calculating the Laplacian. The discrete Laplacian operator on piecewise linear functions over triangulated manifolds is given by:

$$\Delta f_i = \sum_{j \in N_i} W_{ij}(f_j - f_i) \tag{3}$$

where N_i is the set of vertices adjacent to vertex i and W_{ij} is a scalar weight assigned to the directed edge (i, j).

For graphs free of any geometry embedding, it is customary to use the combinatorial weights $W_{ij} = 1/\deg(i)$ in defining the operator. However, for 2-manifold mapped in $\Re^3$, the appropriate choice is a discrete sets of harmonic weights, suggested by Dong [5] and is the one used in this paper (see Equation 4):

$$W_{ij} = \frac{1}{2}(\cot \alpha_{ij} + \cot \beta_{ij}). \tag{4}$$

Here α_{ij} and β_{ij} are the opposite angles to the edge (i, j).

Representing the function f, by the column vector of its values at all vertices $f = [f_1, f_2, \ldots, f_n]^T$, one can reformulate the Laplacian as a matrix $\Delta f = -Lf$ where the Laplacian matrix L each elements are defined by:

$$L_{ij} = \begin{cases} \sum_k W_{ik} & \text{if } i = j, \\ -W_{ij} & \text{if } (i,j) \text{ is an edge of } S, \\ 0 & \text{in other case.} \end{cases} \tag{5}$$

where k is the number of neighbors of the vertex i. The Eigenvalues $\lambda_1 = 0 \leq \lambda_2 \leq \ldots \leq \lambda_n$ of the matrix L forms the *spectrum* of mesh S. Besides describing the square of the frequency and the corresponding eigenvectors $e_1, e_2, \ldots, e_n$ of L, one can define piecewise linear functions over S using progressively higher frequencies [13].

4 Literature Review

There are many research results that deals with fitting surface model on triangular meshes. We will review some of them.

Loop [10] generates B-spline surfaces on irregular meshes. These meshes do not require a known object topology, and therefore, they can be configured arbitrarily without carrying a sequence of the 3D coordinates of the points set. The advantage of this method is that it uses different spline types for the surface approximation. The algorithm was tested using synthetic data with low curvature.

Eck and Hoppe [6] present the first solution to the fitting problem of B-spline surfaces on arbitrary topology surfaces from disperse and unordered points. The method builds an initial parametrization, which in turn is re-parametrized to build a triangular base, which is then used to create a quadrilateral domain. In the quadrilateral domain, the B-spline patches adjust with a continuity degree of C^1. This method, although effective, is quite complex due to the quantity of steps and process required to build the net of B-spline patches on the adjustment surface.

Krishnamurthy and Levoy [9] presented a novel approach to adjust NURBS surface patches on cloud of points. The method consists of building a polygonal mesh on the points set first. Then on this mesh, a re-sampling is performed to generate a regular mesh, on which NURBS surfaces patches can be adjusted. The method has poor performance when dealing with complex surfaces. Other limitations are the impossibility to apply the method to surfaces having holes, and the underlying difficulty to keep continuity on the NURBS surface patches.

Park [12] proposed a two-phase algorithm. In the first phase, a grouping of the points is performed by means of the k-means algorithm to create a polyhedral mesh approximation of the points, which is later reduced to a triangular mesh, on which a quadrilateral mesh is built. In the second phase, the initial model is used to build a net of NURBS patches with continuity C^1. Park's proposal assumes that the cloud-of-points is closed in such a way that the NURBS patches network is fully connected. This implies that the proposed method is not applicable to open surfaces. The use of NURBS patches implies an additional process keeping continuity at the boundary, making the method computationally expensive even when the irregularity of the surface does not require it.

Boulanger *et al.* [3] describe linear approximation of continuous pieces by means of trimmed NURBS surfaces. This method generates triangular meshes which are adaptive to local surface curvature. First, the surface is approximated with hierarchical quadrilaterals without considering the jagged curves. Later, jagged curves are inserted and hierarchical quadrilaterals are triangulated. The result is a triangulation which satisfies a given tolerance. The insertion of jagged curves is improved by organizing the quadrilaterals' hierarchy into a *quad-tree* structure. The quality of triangles is also improved by means of a Delaunay triangulation. Although this method produces good results, it is restricted to surfaces which are continuous and it does not accurately model fine details, limiting its application for objects with an arbitrary topology.

Gregorski [8] proposes an algorithm which decomposes a given points-set into a data structure *strip tree*. The *strip tree* is used to adjust a set of minimal squares quadratic surfaces to the points cloud. An elevation to bicubic surfaces is performed on the quadratic surfaces, and they are merged to form a set of B-spline surfaces which approximates the given points-set. This proposal can not be applied to closed surfaces or surfaces which curve themselves. The proposal is highly complex because it has to perform a degree elevation and a union of patches on B-spline patches at the same time that a continuity degree C^1 is performed among adjacent patches.

Bertram [2] proposes a method to approximate in an adaptive way to disperse points by using triangular hierarchical B-splines. A non-uniform distribution of sampling on the surface is assumed, in such a way that zones with a high curvature present a denser sampling that zones with a low curvature. This proposal uses patches for data adjustment which add quality to the solution.

A different approach is presented by Yvart *et al.* [14], which uses triangular NURBS for dispersed points adjustment. Triangular NURBS do not require that the points-set has a rectangular topology, although it is more complex that NURBS. Similar to previous works, it requires intermediate steps where triangular meshes are reconstructed, re-parametrization processes are performed, and continuity patches G^1 are adjusted to obtain a surface model.

5 Quadrilateralization of Triangular Meshes Using Morse Theory

Representation by means of NURBS patches requires building a regular base on which it is possible to estimate the set of parameters which permit the calculation of surfaces segments which correctly represent every region of the objects. Representation of an object by means of small surface segments, improves the fitting quality in comparison with representation by means of a single continuous surface. Because each segment can be better fitted to local features, which permits modelling small details without excessive loss of surface smoothness.

Construction of a regular base consists of converting a triangular representation of the quadrilaterals set, which permits a complete description of the object's geometry. In general, NURBS surfaces requires a regular base. This does not indicate that the surface is rectangular. However, building a rectangular base permits to easily and directly regularize equidistantly each one of its sides. Because of complex and diverse forms which free form objects can take, obtaining a quadrilateral description of the whole surface is not a trivial problem (See Algorithm 1).

Localizing Critical Points

The procedure proposed at this paper estimates an initial quadrilaterization of the mesh, using spectral analysis by means of the Morse theory. Initially, the quadrilateral's vertexes are obtained as a critical points-set of a Morse

Algorithm 1: Quadrilateralization method of a triangular mesh.

```
Quadrilateralization();
begin
    1. Critical points computation;
    2. Critical points interconnection;
end
```

function. Morse's discrete theory guarantees that, without caring about topological complexity of the surface represented by triangular mesh, a complete quadrilateral description is obtained. That is to say, it is possible to completely divide objects' surfaces by means of rectangles. In this algorithm, an equation system for the Laplacian matrix is solved by calculating a set of eigen-values and eigen-vectors for each matrix (Equation 5).

Morse-Smale Complex is obtained from the connection of a critical points-set which belongs to a field of the Laplacian matrix. The definition of a field of the matrix is obtained by selecting the set of vectors associated to a solution value of the equation. As Morse function represents a function in the mesh, each eigen-value describes the frequency square of each function. Thus, selecting each eigen-value directly indicates the quantity of critical points which the function has. For higher frequency values, a higher number of critical points will be obtained. This permits representing each object with a variable number of surface patches. The eigen value computations assigns function values to every vertex of the mesh, which permits determining whether a vertex of the mesh is at critical points of the Morse function. In addition, according to a value set obtained as the neighborhood of the first ring of every vertex, it is possible to classify the critical points as maximum, minimum or "saddle points." Identification and classification of every critical point permits building the Morse-Smale complex.

Critical points Interconnection

Once critical points are obtained and classified, then they should be connected to form the quadrilateral base of the mesh. The connection of critical points is started by selecting a "saddle point" and by building two inclined ascending lines and two declined descending lines. Inclined lines are formed as a vertex set ending at a maximum critical point. Reversely, a descending line is formed by a vertex path which ends at a minimum critical point. It is allowed to join two paths if both are ascending or descending.

After calculating every paths, the triangulation of K surface is divided into quadrilateral regions which forms Morse-Smale complex cells. Specifically, every quadrilateral of a triangle falls into a "saddle point" without ever crossing a path. The complete procedure is described in Algorithm 2:

Algorithm 2: Bulding method of MS cells.

```
Critical points interconnection();
begin
    Let T={F,E,V} M triangulation;
    Initialize Morse-Smale complex, M=0;
    Initialize the set of cells and paths, P=C=0;
    S=SaddlePointFinding(T);
    S=MultipleSaddlePointsDivission(T);
    SortByInclination(S);
    for every s ∈ S in ascending order do
        CalculeteAscedingPath(P);
    end
    while exists intact f ∈ F do
        GrowingRegion(f, p0, p1, p2, p3);
        CreateMorseCells(C, p0, p1, p2, p3);
    end
    M = MorseCellsConnection(C);
end
```

6 Regularization of the Quadrilateral Mesh

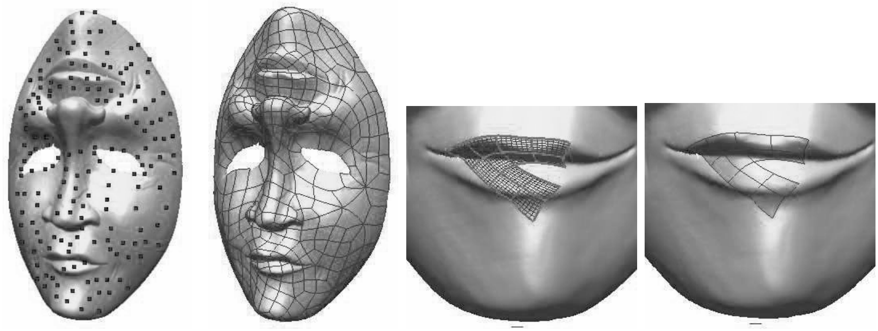
Because the surface needs to be fitted using NURBS patches, it is necessary to regularize the quadrilaterals obtained from the mesh. In Algorithm 3, the proposed method to regularize these quadrilaterals is presented.

Algorithm 3: Quadrilateral mesh regularization method..

```
Regularization();
begin
    1. Quadrilateral selection;
    2. Selection of a border of the selected quadrilateral and its opposite;
    3. Regularization using B-splines with lambda density;
    4. Regularized points match by means of geodetics FMM;
        4.1 Smoothing of geodetic with B-splines;
    5. Points generating for every B-spline line with lambda density;
end
```

One of the quadrilaterals is selected from the mesh, and later a border is selected from each quadrilateral and its opposite. The initially selected border is random. The opposite order is searched as one which does not contain the vertexes of the first one. If the first selected border has vertexes A and B, it is required that the opposite border does not contain vertexes A and B, but the remaining, B and C.

Later, B-splines are fitted on selected borders with a λ density, to guarantee the same points for both borders are chosen, regardless of the distance between them. In general, a B-spline does not interpolate every control point; therefore,



(a) Critical points. (b) Quadrilateralized object.

Fig. 1. Quadrilaterization of a complex curve surface.

they approximate curves which permit a local manipulation of the curve, and they require fewer calculations for coefficient determination.

Having these points at selected borders, it is required to match them. This is done with FMM (*Fast Marching Method*). This algorithm is used to define a distance function from an origin point to the remainder or surface in a magnitude order of $O(nlogn)$. This method integrates a differential equation to obtain the geodetic shortest path by traversing the triangles vertexes.

At the end of the regularization process, B-splines are fitted on geodetic curves and density λ points are generated at every curve which unite the border points of quadrilateral borders, to finally obtain the grid which is used to fit the NURBS surface.

7 Results Analysis

The results obtained at each one of the intermediate stages of the proposed algorithms in this paper are shown in Figure 1. This object called Mask is composed of 84068 points. The reconstruction of the object took an average time of 32 minutes on a dual Opteron PC .

7.1 Comparison Between Branch's Method and Eck and Hoppe's Method

The metric for adjustment error measurement previously described, was used in this thesis to measure the adjustment error for each patch before and after the optimization by means of an evolutive strategy. The obtained results show the effectiveness of the proposed method. During the tests were used $(\mu+\lambda)-ES$ and $(\mu,\lambda)-ES$ evolutionary strategies, but the perform of the $(\mu+\lambda)-ES$ was superior in terms of error reduction. This behavior is because $(\mu+\lambda)-ES$

incorporate information from parents and children to next generation, which maintain the best individuals during the evolutionary process, even though in this way the process can be fall in local minimal. In contrast with $(\mu, \lambda) - ES$ which can forget and so exclude the information of the best individuals. In terms of execution time, $(\mu, \lambda) - ES$ had a negligible advantage over $(\mu + \lambda) - ES$. Only results of the $(\mu + \lambda) - ES$ are provided. During the evolutionary process, the weighting factors are restricted to the $[0, 1]$ interval. If as a results of a mutation, a recombination or a simple initialization, the weighting factors are outer to this interval, its value is set to zero, or set to one, according to the case.

On the other hand, the work by Eck and Hoppe [6] performs the same adjustment by means of a network of B-spline surface patches adaptatively refined until they obtain a given error tolerance. The process of optimization performed by Eck and Hoppe reduces the error by generating new patches, which considerably augments the number of patches which represent the surface. The increment of the number of patches reduces the error because the regions to be adjusted are smaller and more geometrically homogeneous. In the method proposed in this thesis, the optimization process is focused on improving the adjustment for every patch by modifying only its parameterization (control points weight). Because of that, the number of patches does not augment after optimization process. The final number of patches which represent every object is determined by the number of critical points obtained in an eigenvector associated with the eigenvalue (λ) selected from the solution system of the Laplacian matrix, and it does not change at any stage of the process.

Figure 2 contains a couple of objects (foot and skidoo) reported by Eck and Hoppe. Every object is shown triangulated starting with the points cloud. The triangulation is then adjusted with a patch cloud without optimizing and the result obtained after optimization. The adjustment with the method proposed in this thesis, represents each object, with 27 and 25 patches, while Eck and Hoppe use 156 and 94 patches. This represents a reduction of 82% and 73% fewer patches respectively, in our work.

With respect to the reduction of the obtained error in the optimization process in each case, with the proposed method in this thesis, the error reduces an average of 77% (see Table 1), a value obtained in an experimental test with 30 range images (see Table 1). Among these appear the images included in Figure 2. The error reported in Eck and Hoppe for the same images of Figure 2 allow a error reduction of 70%. In spite of this difference which is given between our method with respect to Eck and Hoppe's method, we should emphasize that error metrics are not the same, Eck and Hoppe's method is a measurement of RMS, ours method corresponds to an average of distances of projections of points on the surface.

Another aspect to be considered in the method comparison is the number of patches required to represent the object's surfaces. In Eck's work, the number of patches used to represent the object's increase is an average of 485% in relation to the initial quadrilaterization, while in the method proposed in this thesis, the number of patches to represent the surface without optimization, and the optimized one, is constant.

Image	Initial Error	Optimization Error	Error Reduction(%)
1	19,43	4,41	77,30
2	12,15	2,21	81,81
3	14,25	3,12	78,11
4	13,47	2,74	79,66
5	11,45	2,12	81,48
6	15,26	3,21	78,96
7	14,84	3,14	78,84
8	18,45	3,54	80,81
9	12,12	2,04	83,17
10	14,32	3,31	76,89
11	15,31	3,91	74,47
12	16,70	4,09	75,51
13	20,05	4,48	77,66
14	18,23	4,27	76,58
15	13,24	3,12	76,44
16	19,32	4,45	76,97
17	17,32	4,01	76,85
18	15,24	3,45	77,36
19	16,24	3,69	77,28
20	11,25	2,65	76,44
21	17,32	3,56	79,45
22	14,25	3,25	77,19
23	11,22	2,35	79,06
24	13,26	3,21	75,79
25	14,15	4,25	69,96
26	16,25	4,21	74,09
27	14,25	3,69	74,11
28	18,23	4,56	74,99
29	12,20	2,98	75,57
30	19,43	4,41	77,30
Reduction Average			77,34

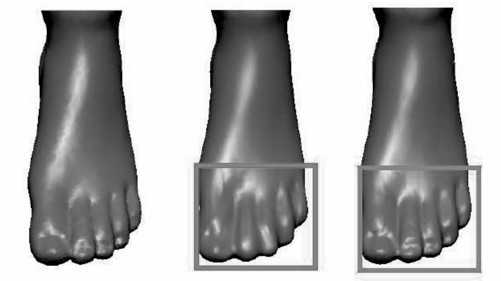
Table 1. Percentage of reduction of the error by means of optimization using our method.

8 Conclusion and Future Work

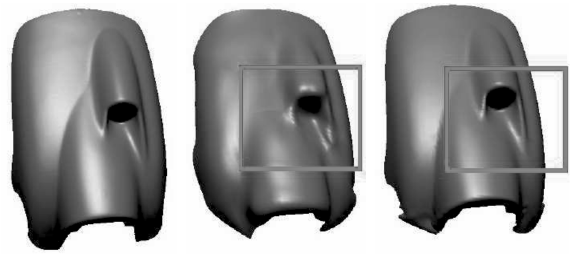
A novel method of quadrilateralization by means of spectral analysis of meshes and Morse theory has been proposed, starting from a triangular mesh. This method is topologically robust and guarantees that the complex base be always quadrilateral, thus avoiding ambiguities between quadrilaterals.

As future work, determination of the quantity of critical points in the following way:

- Quadrangulation refinement to take surface geometrical singularities (to be defined) into account, and optimization in terms of angles in quadrangles.



(a) Triangulated model P(20,021 points) (b) 27 patches model (Branch method) (c) 27 patches model (Branch method)



(d) Triangulated model P(37,974 points) (e) 25 patches model (Branch method) (f) 25 patches model (Branch method)

Fig. 2. Comparison Between Branch's Method and Eck and Hoppe's Method.

- To solve the problem of the determination of the control points, explore the use of nonlinear optimization methods, that can be applied efficiently by means of a parallel implementation.

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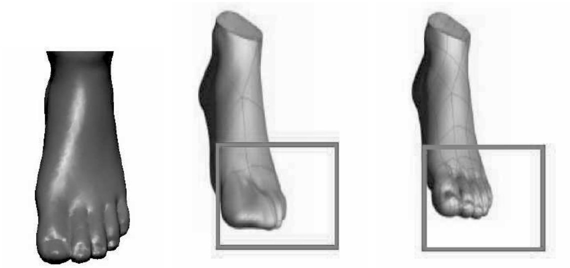
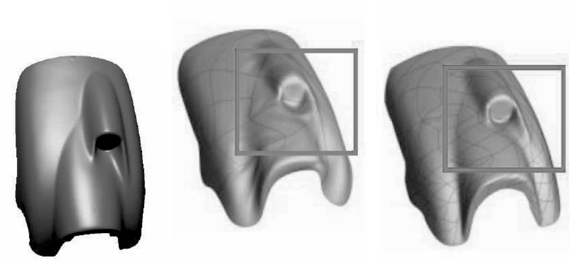


(g) Triangulated model P(20,021 points) (h) 29 patches model (Eck and Hopper's Method) (i) 156 patches model (Eck and Hopper's Method)



(j) Triangulated model P(37,974 points) (k) 15 patches model (Eck and Hopper's Method) (l) 94 patches model (Eck and Hopper's Method)

Fig. 2. Comparison Between Branch's Method and Eck and Hoppe's Method.

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