

Generation of Quasi-optimal Meshes Based on a Posteriori Error Estimates

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Summary. We study methods for recovering tensor metrics from given error estimates and properties of meshes which are quasi-uniform in these metrics. We derive optimal upper and lower bounds for the global error on these meshes. We demonstrate with numerical experiments that the edge-based error estimates are preferable for recovering anisotropic metrics than the conventional element-based errors estimates.

1 Introduction

Challenging problems in predictive numerical simulation of complex systems require solution of many coupled PDEs. This objective is difficult to achieve even with modern parallel computers with thousand processors when the underlying mesh is quasi-uniform. One of the possible solutions is to optimize the computation mesh. The physically motivated optimization methods require error estimates and a space metric related to these estimates. In this article, we show that the quasi-optimal meshes or equivalently the optimal metric can be recovered from the edge-based error estimates.

The metric-based mesh adaptation is the most powerful tool for optimal mesh generation. It has been studied in numerous articles and books (see, e.g. [2, 7, 1, 6] and references therein). However, theoretical study of optimal meshes is relatively new area of research [1, 13, 15, 4, 3, 8, 11]. An optimal metric may be derived from a *posteriori* error estimates and/or solution features. Also, metric modification is a simple way to control mesh properties. Eigenvalues and eigenvectors of the tensor metric allow to control the shape and orientation of simplexes. The impact of metric modification on error estimates has been studied in [15].

In this paper, we study piecewise constant metrics recovered from edge-based error estimates and properties of meshes which are quasi-uniform in these metrics. We prove that these meshes are quasi-optimal, in a sense that

the global error is bounded from above and below by $|\Omega|_M^2 N_T^{-1}$, where N_T is the number of simplexes and $|\Omega|_M$ is the volume of computational domain in metric M .

We study numerically three methods for recovering a continuous piecewise linear metric from given error estimates. In the first method, we take edge-based error estimates and combine ideas from [14] and [1] to build an anisotropic metrics. In the second method, we use again edge-based error estimates and the least square approach to build another anisotropic metric. Recently, we found that the least square approach has been already employed in [5] where a particular edge-based error estimates are derived. In our method, supported by Theorem 1, we may use any edge-based error estimates which makes it more general. Note that the metric generated with the least-square approach results in less sharper estimates.

Our experience shows that simple methods like the least square approach, applied to element-based error estimates, result frequently in isotropic metrics even for anisotropic solutions. Therefore, in the third method, we use element-based error estimates and the ZZ-interpolation method to generate a metric. This metric is inherently isotropic and used only for comparison purposes. We also show that there are cases when the element-based error estimates provide *no* information about an anisotropic solution, while the edge-based estimates allow to generate optimal anisotropic metrics.

The article outline is as follows. In Section 2, we recall basic concepts of metric-based mesh adaptation. In Section 3, we present the main result for piecewise constant metrics. In Section 4, we consider methods for recovering continuous metrics. In Section 5, we illustrate our findings with numerical experiments.

2 Metric-based mesh generation

Let Ω be a polygonal domain in \mathbb{R}^2 , and $M(x)$ be a symmetric positive definite 2×2 matrix (tensor metric) defined for every $x = (x_1, x_2)$ in Ω . In metric M , the volume of a domain $D \subset \Omega$ and the length of a curve ℓ are defined as follows:

$$|D|_M = \int_D \sqrt{\det M(x)} \, dx, \quad |\ell|_M = \int_0^1 \sqrt{(M(\gamma(t))\gamma'(t), \gamma'(t))} \, dt, \quad (1)$$

where $\gamma(t) : [0, 1] \rightarrow \mathbb{R}^2$ is a parametrization of ℓ .

Let Ω_h be a conformal triangular partition (triangulation) of Ω . There are a number of ways to generate a triangulation which is quasi-uniform in metric $M(x)$ (see, for example, [7, 1]). One of the robust metric-based mesh generation methods uses a sequence of *local* mesh modifications [2, 1]. The list of mesh modifications includes alternation of topology with node deletion/insertion, edge swapping, and node movement. The topological operation is accepted if

it increases a mesh quality which is a measure of mesh M -quasi-uniformity. Different mesh qualities may be used [2, 1, 12, 9].

It has been shown in [1, 12, 13] that for a particular choice of metric M , the M -quasi-uniform meshes provide the same asymptotic reduction of the piecewise linear interpolation error as the optimal mesh. The objective of this paper is to discuss how such a metric may be generated based on robust and reliable a posteriori error estimates.

3 A posteriori error estimates and mesh quasi-optimality

In this section we assume that a robust and reliable *a posteriori* error estimate η_e to a true error ξ_e is assigned to every mesh edge e :

$$C_1 \xi_e \leq \eta_e \leq C_2 \xi_e \quad (2)$$

and that constants C_1, C_2 do not depend on Ω_h . The estimates of these type are not popular in scientific computations since they measure error in unusual way (on mesh edges). However, we show below that these estimates are more preferable for building anisotropic meshes.

For every triangle Δ with edges e_1, e_2, e_3 , we define a constant (tensor) metric M_Δ which satisfies the following equations:

$$(M_\Delta e_i, e_i) = \eta_{e_i}, \quad i = 1, 2, 3. \quad (3)$$

Hereafter, we use e both for a mesh edge and for a vector from one edge-end point to the other. Formula (3) implies that entries of matrix

$$M_\Delta = \begin{pmatrix} \mathbf{m}_{11} & \mathbf{m}_{12} \\ \mathbf{m}_{12} & \mathbf{m}_{22} \end{pmatrix}$$

are the unknowns of the linear system:

$$(\mathbf{m}_{11}e_{i,x} + \mathbf{m}_{12}e_{i,y})e_{i,x} + (\mathbf{m}_{12}e_{i,x} + \mathbf{m}_{22}e_{i,y})e_{i,y} = \eta_{e_i}, \quad i = 1, 2, 3,$$

where $e_i = (e_{i,x}, e_{i,y})^T$. This linear system is non-singular since its determinant is nonzero and equals to $16|\Delta|^3$ [14], where $|\Delta|$ denotes the area of Δ . Let $|M_\Delta|$ be the spectral module of M_Δ . The metric M is defined as the piecewise constant metric on Ω_h with values $|M_\Delta|$.

For simplicity, we assume that $\det(M_\Delta) \neq 0$. Otherwise, we must introduce a perturbation of zero eigenvalues of M_Δ [1] and take it into account in the analysis.

For every element Δ , we introduce the secondary error estimator:

$$\chi_\Delta^2 = \sum_{e \subset \partial\Delta} \eta_e^2$$

and the global error estimator

$$\chi^2 = \frac{1}{3} \sum_{\Delta \in \Omega_h} \chi_\Delta^2.$$

Using the definition of edge length in the constant metric $|M_\Delta|$,

$$|e|_{|M_\Delta|}^2 = (|M_\Delta|e, e),$$

and the inequality

$$\max_{e \in \partial\Delta} (M_\Delta e, e)^2 \geq C|\Delta|_{|M_\Delta|}^2,$$

proved in [4], we can estimate χ_Δ^2 from both sides:

$$\chi_\Delta^2 = \sum_{e \in \partial\Delta} (M_\Delta e, e)^2 \leq \sum_{e \in \partial\Delta} (|M_\Delta|e, e)^2 \leq |\partial\Delta|_{|M_\Delta|}^4$$

and

$$\chi_\Delta^2 = \sum_{e \in \partial\Delta} (M_\Delta e, e)^2 \geq \max_{e \in \partial\Delta} (M_\Delta e, e)^2 \geq C|\Delta|_{|M_\Delta|}^2.$$

Here $|\partial\Delta|_{|M_\Delta|}$ is the perimeter of Δ measured in metric $|M_\Delta|$.

Assume that triangulation Ω_h with N_T triangles is M -quasi-uniform, i.e., for any Δ in Ω_h we have

$$|\partial\Delta|_M^2 \simeq |\Delta|_M \quad \text{and} \quad |\Delta|_M \simeq N_T^{-1}|\Omega|_M.$$

Then

$$\chi^2 \leq \frac{1}{3} \sum_{\Delta \in \Omega_h} |\partial\Delta|_{|M_\Delta|}^4 \leq C \sum_{\Delta \in \Omega_h} |\Delta|_{|M_\Delta|}^2 \leq CN_T^{-1}|\Omega|_M^2$$

and

$$\chi^2 \geq C \sum_{\Delta \in \Omega_h} |\Delta|_{|M_\Delta|}^2 \geq CN_T \min_{\Delta \in \Omega_h} |\Delta|_{|M_\Delta|}^2 \geq CN_T^{-1}|\Omega|_M^2.$$

We proved the following result.

Theorem 1. *For any edge e , let η_e be a given a posteriori error estimator satisfying (2). Let the piecewise constant metric M be defined by (3) and $|M_\Delta|$. If Ω_h is a M -quasi-uniform triangulation with N_T triangles, then it is quasi-optimal:*

$$cN_T^{-1}|\Omega|_M^2 \leq \sum_{\Delta \in \Omega_h} \sum_{e \in \partial\Delta} \xi_e^2 \leq CN_T^{-1}|\Omega|_M^2$$

with constants c, C independent of N_T and Ω_h .

The theorem holds for any definition of error ξ_e . In reality, the sum of all ξ_e^2 represents some norm of error, $\|u - u_h\|_{*,\Omega}^2$, where u is the exact solution, and u_h is its approximation. For instance, this is true if ξ_e^2 is proportional to $|\sigma_e|$ where σ_e is the union of triangles sharing e .

The global error on the M -quasi-uniform mesh is also the sum of approximately equal element-based errors $\chi_\Delta^2 \approx |\Delta|_{|M_\Delta|}^2$. However, it is not clear how to use only these element-based error estimates (without using η_e^2) to recover an anisotropic metric (see also discussion at the end of Section 4). Finally, we emphasize that the N_T^{-1} -asymptotic error reduction holds on anisotropic meshes as long as $|\Omega|_M$ is independent of N_T .

To produce a quasi-optimal mesh, we suggest the following adaptive algorithm.

Initialization Step. Generate an initial triangulation Ω_h . Set $\chi = +\infty$. Choose the final number N_T of mesh elements.

Iterative Step.

1. Compute the approximate solution u_h .
2. Compute the estimators η_e and χ . Stop, if χ is not reduced.
3. Otherwise, compute metric M from η_e .
4. Generate a M -quasi-uniform mesh $\tilde{\Omega}_h$ with N_T elements.
5. Set $\Omega_h := \tilde{\Omega}_h$ and go to step 1.

This iterative method requires an initial mesh which may be arbitrary and very coarse.

4 Recovery of a continuous metric

Our experience shows that continuous metrics are more beneficial in the adaptive metric-based generation. To this end, we suggest a simple technique to generate a continuous piecewise linear metric \widehat{M} first by recovering it at mesh nodes and then by interpolating it linearly inside mesh elements.

We consider three methods to recover \widehat{M} at mesh nodes. First, for a mesh node a_i we define the nodal *tensor* metric \widehat{M}_i by taking $|M_\Delta|$ from one of the surrounding elements:

$$\widehat{M}_i = \arg \max_{|M_\Delta|, a_i \in \Delta} \det(|M_\Delta|). \quad (4)$$

Second, for a mesh node a_i , the nodal *tensor* metric \widehat{M}_i is defined using η_e on mesh edges e incident to a_i . Let κ_i be the number of these edges. As in (3), we would like to have

$$(\widehat{M}_i e_j, e_j) = \eta_{e_j}, \quad j = 1, \dots, \kappa_i. \quad (5)$$

In algebraic terms (5) is a linear system

$$Am = b$$

where $m \in \mathfrak{R}^3$ is the vector of unknown entries of matrix \widehat{M}_i , and $A \in \mathfrak{R}^{\kappa_i \times 3}$. For $\kappa_i > 3$ this system may be overdetermined. In this case, we use the least squares solution:

$$m = \arg \min_m \|Am - b\|^2. \quad (6)$$

We have learned recently that the least square techniques has been used in [5]. The authors use the discrete solution and its gradient to generate vector b . In our method, supported by Theorem 1, we may use any edge-based error estimates which makes it more general.

Third, we recover a *scalar* (isotropic) metric at each triangle based on a given element-based error estimate η_Δ :

$$M_\Delta = |\Delta|^{-1} \eta_\Delta I.$$

Let \bar{M} be a piecewise constant scalar metric composed of M_Δ . Note that arguments of Theorem 1 may be applied to \bar{M} -quasi-uniform triangulations with N_T elements if

$$C_1 \xi_\Delta \leq \eta_\Delta \leq C_2 \xi_\Delta,$$

where ξ_Δ is the true element-based error. The definition (1) implies that $|\Delta|_{M_\Delta}^2 = |\Delta|^2 \det(M) = \eta_\Delta^2$. Thus,

$$\sum_{\Delta \subset \Omega_h} \xi_\Delta^2 \simeq \sum_{\Delta \subset \Omega_h} \eta_\Delta^2 = \sum_{\Delta \subset \Omega_h} |\Delta|_{M_\Delta}^2 \simeq N_T^{-1} |\Omega|_{\bar{M}}^2.$$

However, these shape-regular meshes will be quasi-optimal if and only if $|\Omega|_{\bar{M}} \simeq |\Omega|_M$, that is, if the optimal mesh is shape-regular. For anisotropic solutions, $|\Omega|_{\bar{M}}$ will depend on the number of simplexes.

In order to define a nodal *scalar* (isotropic) metric from the element-based metric M , we consider triangles with a common mesh node a_i and construct the best (in terms of the least squares) linear function which approximates given errors at triangle centers. The value of the linear function at node a_i defines the scalar metric \widehat{M}_i . This type of interpolation is known as the ZZ-interpolation method [16]. The use of scalar metrics represents a group of methods which generate adaptive meshes using a size function.

Let us show that there are cases when the element-based error estimates provide no information about solution anisotropy. Let $u(x_1, x_2) = x_1^2$ and

$$\eta_\Delta = \int_\Delta (u - u_I)^2 dx$$

where u_I is the continuous piecewise linear interpolant of u on a structured triangular mesh. The triangular mesh is obtained from a square mesh by dividing each square into two triangles. The direct calculations show that η_Δ is the same for all elements. On the other hand, $\eta_e = 0$ on vertical edges which indicates the direction of solution anisotropy. More complicated examples are consider in the next section.

5 Numerical experiments

In this section we consider a model problem of piecewise linear interpolation of an anisotropic function and three nodal metrics. For the first two metrics, we assume that the edge-based error estimates η_e are given. The anisotropic metric $\widehat{M}^{(1)}$ is recovered by (3) and (4). The anisotropic metric $\widehat{M}^{(2)}$ is recovered by (5) and (6). The isotropic metric $\widehat{M}^{(3)}$ is defined with the ZZ-interpolation method using given element-based error estimates η_Δ .

To generate a \widehat{M} -quasi-uniform mesh, we use the algorithms described in articles [1, 10] and implemented in the software package `ani2d` (<http://sourceforge.net/projects/ani2d>). In all experiments, we start with a quasi-uniform unstructured mesh with 2792 triangles. We request that the final adaptive mesh must have 1000 triangles. Depending on the mesh quality, the final number of elements in the quasi-optimal mesh may deviate from the requested number [10]. In the experiments, the target mesh quality was 0.7 on a scale from 0 to 1, where quality 1 corresponds to an ideal mesh.

Let us consider the function

$$u(x_1, x_2) = \exp(2(x_1^\alpha + x_2^\alpha)), \quad \alpha > 0,$$

in the unit square. The function is constant along curves $x_1^\alpha + x_2^\alpha = \text{const}$ and becomes one-dimensional when $\alpha \rightarrow 1$. We expect that the optimal mesh be shape-regular for $\alpha = 2$ and strongly stretched for $\alpha = 1.01$. For simplicity, we choose the edge-based, η_e , and element-based, η_Δ , estimators as the properly scaled exact interpolation errors:

$$\eta_e^2 = \max_{(x_1, x_2) \in e} |u - u_I|^2 |\sigma_e|, \quad \eta_\Delta^2 = \max_{(x_1, x_2) \in \Delta} |u - u_I|^2 |\Delta|.$$

We consider two cases, $\alpha = 1.01$ (see Fig. 1) and $\alpha = 2$ (see Fig. 2). For both values of α , the adaptive iterative method converged in 2 iterations for all metrics. According to Table 1, the tensor metrics results in sharper estimates for the interpolation error. The mean interpolation error on the anisotropic meshes is 10 times smaller than on the isotropic mesh. Fig. 2 shows that the shape-regular quasi-optimal mesh corresponding to $\widehat{M}^{(1)}$ is aligned better with the function u .

Table 1. Mean interpolation errors for different metrics.

α	metric $\widehat{M}^{(1)}$	metric $\widehat{M}^{(2)}$	metric $\widehat{M}^{(3)}$
2.00	1.48e-2	1.54e-2	3.38e-2
1.01	1.99e-3	2.08e-3	2.25e-2

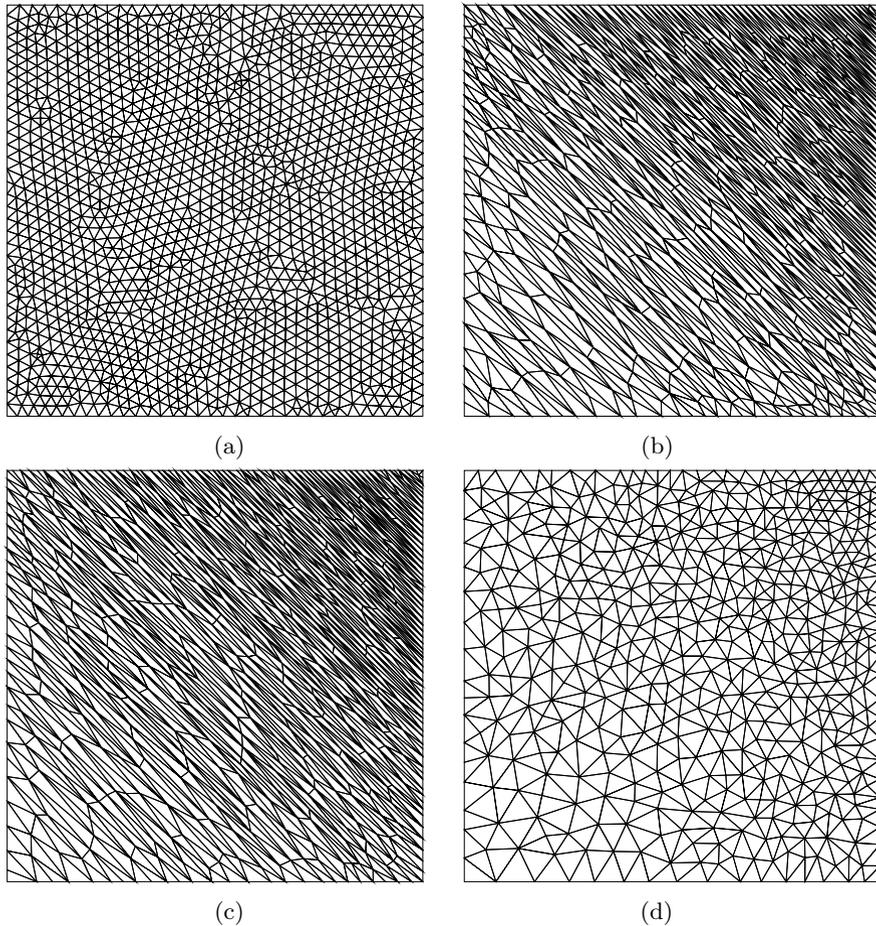


Fig. 1. Adaptive meshes for anisotropic function with $\alpha = 1.01$: (a) Initial mesh with 2792 triangles. (b) $\widehat{M}^{(1)}$ -quasi-uniform mesh with 1042 triangles. (c) $\widehat{M}^{(2)}$ -quasi-uniform mesh with 1018 triangles. (d) $\widehat{M}^{(3)}$ -quasi-uniform mesh with 1006 triangles.

Conclusion

We showed that the robust and reliable edge-based *a posteriori* error estimates may be used for generation of quasi-optimal (possibly anisotropic) meshes. We considered three methods for recovering a continuous piecewise linear tensor

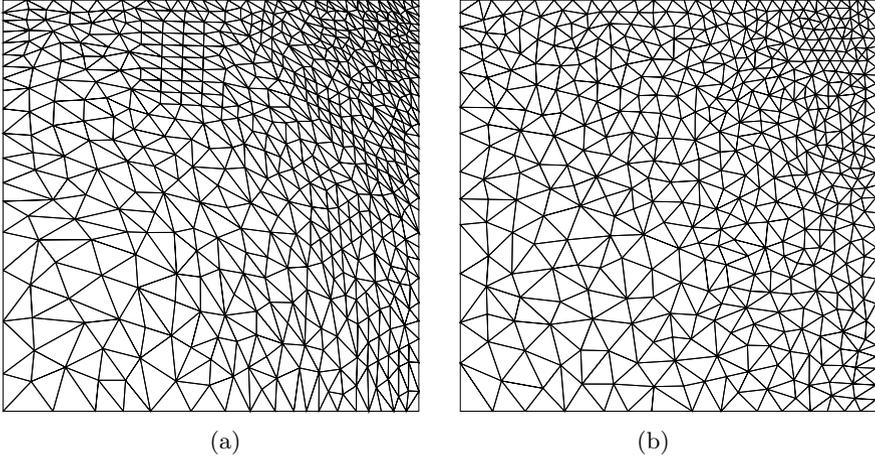


Fig. 2. Adaptive meshes for isotropic function with $\alpha = 2$: (a) $\widehat{M}^{(2)}$ -quasi-uniform mesh with 1073 triangles. (d) $\widehat{M}^{(3)}$ -quasi-uniform mesh with 941 triangles.

metric and demonstrated with numerical experiments efficiency of two of them for generating meshes which minimize the interpolation error.

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