Structured Grid Generation over NURBS and Facetted Surface Patches by Reparametrization

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Abstract

This paper deals with structured grid generation using Floater's parametrization algorithm for surface triangulation. It gives an outline of the algorithm in the context of structured grid generation. Then it explains how the algorithm can be used to generate a structured grid over a singular NURBS surface patch. This is an alternate method to the known carpeting method of reparametrization for structured grid generation over a NURBS surface patch. The paper also explains how to generate a structured grid over a four sided trimmed patch of a facetted surface using the parametrization algorithm. All the procedures are explained using examples.

Keywords: structured grid generation, parametrization, reparametrization, Floater's algorithm, NURBS surface, facetted surface

1 Introduction

Currently common methods of structured grid generation are transfinite interpolation (TFI) and elliptic smoothing [1]. Most of the time, the grid is generated over NURBS or other parametrically defined piecewise smooth surfaces. Direct implementations of TFI and elliptic smoothing work well for simple four-sided surface patches defined over a single surface without singular points — a point on a surface is called a singular point if a directional derivative or normal is zero at that point. To obtain a structured grid over a complex surface patch that is either defined over multiple surfaces or over a surface with singular points, one has to first reparametrize that surface patch by a new surface and then has to generate the grid. The grid points will be finally projected onto the original surfaces if the reparametrization is not exact. One method of reparametrization used by grid generators is by defining a carpet surface [1, 2]. A carpet surface is created by obtaining the four boundary curves, generating a surface from these four curves by TFI, and then projecting this surface onto the underlying original surface patches [1].

In this paper an alternate method of reparametrization is shown for generating structured grids over complex NURBS surface patches. The method also works for generating structured grids over facetted surfaces, which are being used in bio-medical and other applications. The reparametrization is based on Floater’s parametrization algorithm [3] for surface triangulations.

Floater’s algorithm is widely used in computer graphics applications, but it is not used by many people to generate quality grids for Computational Fluid Dynamics (CFD) and Computational Solid Mechanics (CSM) applications. It has been mentioned in [4] as a method of obtaining valid structured grids in the parametric domain of a surface patch, which will then be optimized to obtain quality grids. However that procedure may not work if the parametric function has singularities. Whereas the method proposed in this paper obtains a structured grid directly over the surface patch.

The reparametrization method is being implemented as part of GGTK – the Geometry and Grid Toolkit [5] developed at the University of Alabama at Birmingham. It is being provided in addition to the existing TFI and elliptic methods. In the following sections, the method is explained and illustrated using examples. Section 2 gives an outline of Floater's parametrization algorithm. Section 3 shows how it can be used for structured grid generation over a trimmed NURBS surface patch. Section 4 shows how it can be used to generate structured grids over facetted surfaces. Finally, Section 5 gives conclusions and mentions further work along this direction.

2 Parametrization Algorithm

This section first explains the notations and terminology used in parametrization and then gives an outline of Floater’s parametrization algorithm. It also briefly talks about choice of coordinate values and shape preserving parametrization. Then it mentions how to obtain a structured grid over the surface using the parametrization.

2.1 Notations and Definitions

In [3] a surface triangulation is defined using graph theory notations. But in this paper it will be defined in terms of common notations used in grid generation as follows.

A *surface triangulation S* consists of a set of vertices *V*, a set of edges *E*, and a set of triangular faces *F* satisfying the following properties:

- Each vertex in *V* is a point in R^3; each edge in *E* is a line segment joining two vertices; each triangle is formed by three edges.
- Each vertex will be part of at least one edge and each edge will be part of at least one triangle.
- Intersection of any two triangles will be either empty or an edge or a vertex.

An edge of a surface triangulation *S* is said to be a *boundary edge* if it is shared by only one triangle. A vertex is said to be a *boundary vertex* if it lies on a boundary edge; otherwise it is called an *internal vertex*.

A surface triangulation is said to be *simply connected* if all its boundary edges form a single connected loop. Figure 1 (a) shows a simply connected surface triangulation.

Two surface triangulations are said to be *isomorphic* [3] if there is a one-one correspondence between their vertices, edges and faces in such a way that corresponding edges join corresponding vertices and corresponding triangles are formed by corresponding edges.

A *planar triangulation* is a special case of surface triangulation when all the vertices lie on the *xy* plane R^2.

A surface triangulation *S* is said to have a *parametrization* if there is a planar triangulation that is isomorphic to *S*. Figure 1 (b) shows a planar triangulation that is isomorphic to the surface triangulation shown in Figure 1 (a). That is, Figure 1 (b) is a parametrization of Figure 1 (a).

In [3] it is shown that every simply connected surface triangulation has a parametrization.

2.2 Floater's Algorithm

The procedure given in [3] to compute a planar triangulation that is isomorphic to a given simply connected surface triangulation *S* can be outlined as follows:

Step 1: Let $V = \{P_i\}$ be the set of vertices of *S*. Label the vertices in such a way that $\{P_1, \ldots P_n\}$ are internal vertices and $\{P_{n+1}, \ldots P_N\}$ are boundary vertices, where N is the total number of vertices and n is the number of internal vertices.

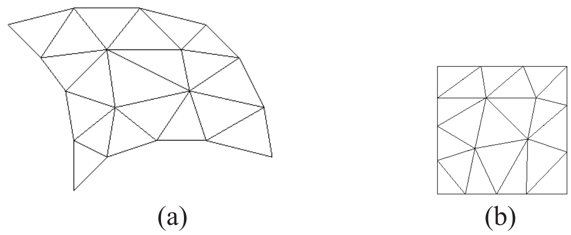


Fig. 1. (a) A simply connected surface triangulation; (b) A planar triangulation isomorphic to (a).

Step 2: Select a convex domain D in R^2 and choose points $Q_{n+1}, \ldots Q_N$ on the boundary of D such that they form an anticlockwise sequence.

Step 3: For each i in $\{1, \ldots, n\}$ and j in $\{1, \ldots, N\}$, choose λ_{ij} such that

$\lambda_{ij} = 0$ if P_iP_j is not an edge

$\lambda_{ij} > 0$ if P_iP_j is an edge

For each i, $\sum_{j=1}^{N} \lambda_{ij} = 1$.

Step 4: Form the system of linear equations

$$Q_i = \sum_{j=1}^{N} \lambda_{ij} Q_j, \; i = 1, \ldots, n,$$

where $Q_1, \ldots Q_n$ are the unknown interior points to be computed in the domain D and $Q_{n+1}, \ldots Q_N$ are the known boundary points chosen in Step 2. The above equation can be rewritten as

$$Q_i - \sum_{j=1}^{n} \lambda_{ij} Q_j = \sum_{j=n+1}^{N} \lambda_{ij} Q_j, \; i = 1, \ldots, n.$$

This can be written in matrix form as **AQ** = **B** with

$$\mathbf{A} = \begin{bmatrix} a_{11} & .. & a_{1n} \\ : & :: & : \\ a_{n1} & .. & a_{nn} \end{bmatrix}, \; \mathbf{Q} = \begin{bmatrix} Q_1 \\ : \\ Q_n \end{bmatrix}, \; \mathbf{B} = \begin{bmatrix} B_1 \\ : \\ B_n \end{bmatrix},$$

where $a_{ii} = 1, \; a_{ij} = -\lambda_{ij}$ for $i \neq j$, $B_i = \sum_{j=n+1}^{N} \lambda_{ij} Q_j$.

Then the matrix **A** is non-singular [3].

Step 5: Solve the above system and obtain the points $Q_1, \ldots Q_n$ in the interior of the domain D.

Step 6: Form a planar triangulation T over D with $\{Q_1, \ldots Q_N\}$ as the set of vertices and having edges and triangles in such a way that Q_iQ_j is an edge in T if and only if P_iP_j is an edge in the given surface triangulation S and $Q_iQ_jQ_k$ is a triangle in T if and only if $P_iP_jP_k$ is a triangle in S.

The planar triangulation T formed as above is a valid triangulation [3] and is isomorphic to the given surface triangulation S. That is, it defines a parametrization of S. For example, the planar triangulation shown in Figure 1 (b) has been obtained by the above procedure as the parametrization of the surface triangulation of Figure 1 (a).

2.3 Choice of λ_{ij}

The values λ_{ij} chosen in Step 3 above are called coordinates of the vertex P_i with respect to its surrounding vertices. The quality of the parametrization will depend on the choice of these coordinates λ_{ij}. In [3] three methods of computing λ_{ij} are discussed resulting in three kinds of parametrization. They are:

1. Uniform parametrization: $\lambda_{ij} = 1/d_i$, where d_i is number of vertices connected to P_i.
2. Weighted least squares of edge lengths: λ_{ij} is proportional to $1/\|P_i - P_j\|$.
3. Shape preserving parametrization: A procedure is given in [3] to compute λ_{ij} that will result in a shape preserving parametrization analogous to the chord length parametrization of a sequence of points.

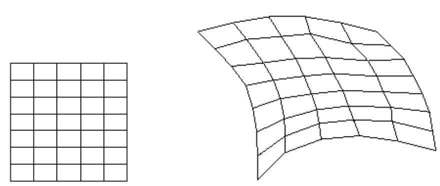
Another choice for λ_{ij} is the mean value coordinates given in [6].

Our purpose of parametrization is to obtain good quality grids on the surface by mapping good quality grids in the new parametric domain D. Hence we need to choose a parametrization that maps good triangles in the parametric domain to good triangles on the surface. This is the characterizing property of the shape preserving parametrization. In the case of uniform parametrization, whatever the shapes of the triangles in the original surface triangulation, the shapes of the triangles of the parametrization in the domain D are good because they satisfy the Laplacian smoothing equation. That means a good triangle in the parametric domain may be mapped to a bad triangle on the surface if we use uniform parametrization. In [3], it is mentioned that the shape preserving parametrization gives smoother results than the other two. Hence in all results shown in this paper, the shape preserving parametrization has been used.

2.4 Structured Grid Generation

Suppose the surface triangulation *S* has four corner points. In order to generate a structured grid over *S*, choose a square or a rectangular domain *D* in Step 2 and choose boundary points in *D* in such a way that the four corner points of *S* are mapped to the four corners of *D*. Then compute the isometric planar triangulation over *D*. After that, generate the desired structured grid over *D*, and map that grid onto *S* using the parametrization map defined as follows: If *Q* is a point in *D* then *Q* will lie in a triangle $Q_iQ_jQ_k$ of the planar triangulation. Let α, β, γ be the barycentric coordinates of Q with respect to $Q_iQ_jQ_k$. Then *Q* will be mapped to *P* on the surface triangulation, where *P* is the point on the triangle $P_iP_jP_k$ with barycentric coordinates α, β, γ.

For the example parametrization shown in Figure 1, Figure 2 (a) shows a structured grid over the parametric domain *D* and Figure 2 (b) shows the resulting structured grid over the surface *S*.



(a) (b)

Fig. 2. (a) Structured grid over the parametric domain D; (b) Corresponding structured grid over the surface S.

2.5 Computational Issues

The parametrization procedure involves solving a linear system of equations. For small values of *n* the system can be solved by LU decomposition. Note that the matrix **A** (obtained in Step 4 of the procedure) is diagonally dominant and the matrix elements a_{ij} have the property that $a_{ij} = 0$

if P_iP_j is not an edge. That is, the matrix **A** is of the same kind as a typical finite element matrix. So, for large values of *n*, one should use other efficient solvers used in finite element methods.

3 Structured Grid over a NURBS Surface Patch

This section shows how to generate a structured grid over a singular NURBS patch using the parametrization described above. The procedure involves two steps: (a) Obtain a surface triangulation over the patch; (b) compute a parametrization using a square domain, and generate the structured grid. These steps are explained below using an example.

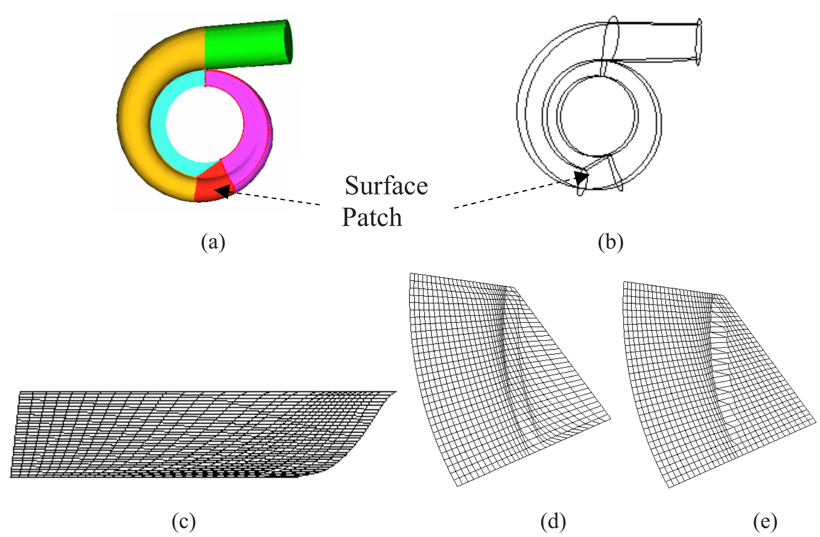
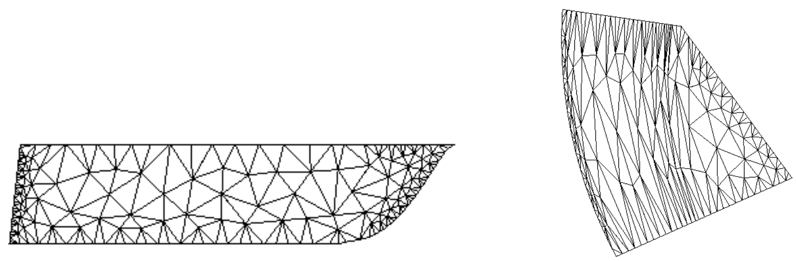


Fig. 3. Singular surface patch of a volute. (a), (b) The patch shown as part of the volute; (c) TFI grid over the trimmed parametric domain of the patch; (d) Corresponding structured grid over the face; (e) Grid obtained by TFI in 3D and projection.

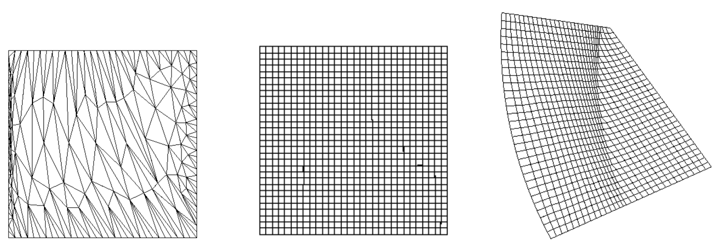
3.1 Example of a singular NURBS patch

An example of a singular NURBS surface patch is shown in Figure 3. The surface patch is one of 26 boundary faces of a *volute model,* shown in Figures 3 (a) and 3 (b). The surface patch is defined by a trimmed NURBS surface [7]. A TFI grid over the trimmed parametric domain of the surface patch is shown in Figure 3 (c). The corresponding TFI grid over the surface patch obtained by the NURBS mapping is shown in Figure 3 (d). Figure 3 (e) shows the grid obtained by TFI in 3D and then projection on to the surface. As can be seen in the Figures 3 (d) and 3 (e), the TFI grids over the surface are not good. This is because the NURBS function has singularities along the line of bending in the interior of the surface. Actually one directional derivative vanishes, and the second order derivative in that direction is discontinuous.



(a) (b)

Fig. 4. (a) Triangulation of the original parametric domain; (b) Corresponding triangulation of the surface patch.



(a) (b) (c)

Fig. 5. (a) Triangulation of the new parametric domain obtained by parametrization of the surface triangulation shown in Fig 4 (b); (b) Structured grid on the new parametric domain; (c) Corresponding structured grid on the surface obtained using the new parametrization mapping.

3.2 Surface Triangulation

To obtain a triangulation of the surface patch, the boundary is first discretized and the corresponding grid points are obtained in the original parametric domain of the NURBS surface. Care is taken to ensure that the set of boundary grid points contains the four corner points of the patch. The parametric domain of the patch is triangulated using an automatic 2D triangulation routine such as *Triangle* [8] that preserves the boundary grid points and adds new interior grid points. The parametric domain triangulation is then mapped on to the surface using the NURBS function. This will result in a surface triangulation S of the surface patch.

Figure 4 (a) shows a triangulation of the parametric domain of the example surface patch and Figure 4 (b) shows the corresponding surface triangulation on the patch. Note that the quality of the surface triangulation is not important for the reparametrization.

3.3 Reparametrization and Structured Grid

For the surface triangulation *S* computed as above, a shape preserving parametrization is computed using the Floater's algorithm explained in Section 2. Since the purpose is to compute a structured grid, a square is chosen as the domain *D* for the parametrization. This will result in an isometric triangulation over *D* and will define a map from *D* on to *S*. In order to obtain a structured grid on the surface patch, corresponding structured grid is generated on the square parametric domain *D* and mapped onto the surface triangulation *S*. The grid points will lie on the triangles of *S* and may not lie on the original surface. So they are finally projected onto the original surface.

For the example surface patch, Figure 5 (a) shows triangulation of the new parametric domain obtained by the parametrization; Figure 5 (b) shows a structured grid on the new parametric domain; Figure 5 (c) shows the final structured grid on the surface patch obtained by mapping the structured grid on the new parametric domain onto the surface triangulation using the parametrization and then projection.

4 Structured Grid Over a Faceted Surface Patch

This section shows how to generate a structured grid over a portion of a facetted surface. A facetted surface is a surface defined by a collection of polygons. They are used to represent surfaces reconstructed from scanned data of objects such as MRI/CT scan of internal human body parts [9]. In this paper it is assumed that a facetted surface is defined by a surface triangulation, that is, it is collection of triangles. Figure 6 shows a human lung model defined as facetted surface consisting of triangles.

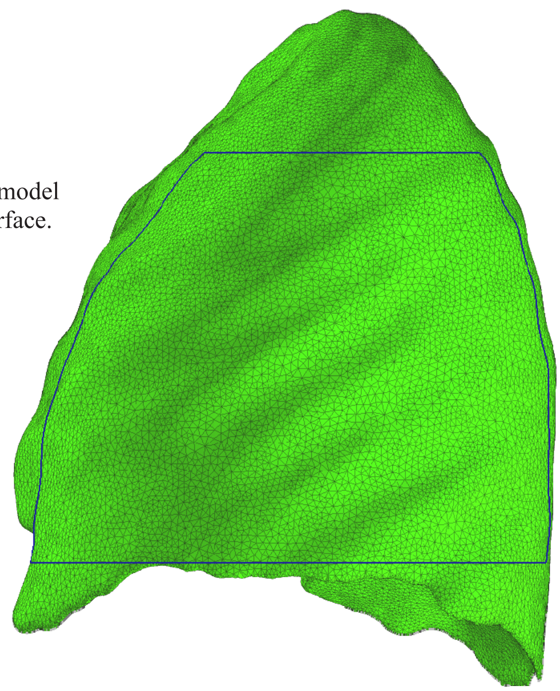


Fig. 6. Human lung model defined by a facetted surface.

4.1 Surface Triangulation

The main issue in generating a structured grid over a facetted surface is defining a four sided surface patch and obtaining a surface triangulation for the surface patch. In many cases, a facetted surface may not have boundary curves on the surface defining four sided patches. In such cases, one can define the required curves by creating curves in 3D space and projecting them onto the surface or by intersecting the surface with planes. These curves may be intersecting the triangles. So in order to obtain surface triangulation of the required patch, one has to retriangulate the surface along these curves and ensure that the curves pass along the edges of the triangulation. Then a surface triangulation of the patch can be obtained by collecting all the triangles inside the patch. Figure 7 shows a surface triangulation over a four sided patch of the lung model obtained as above.

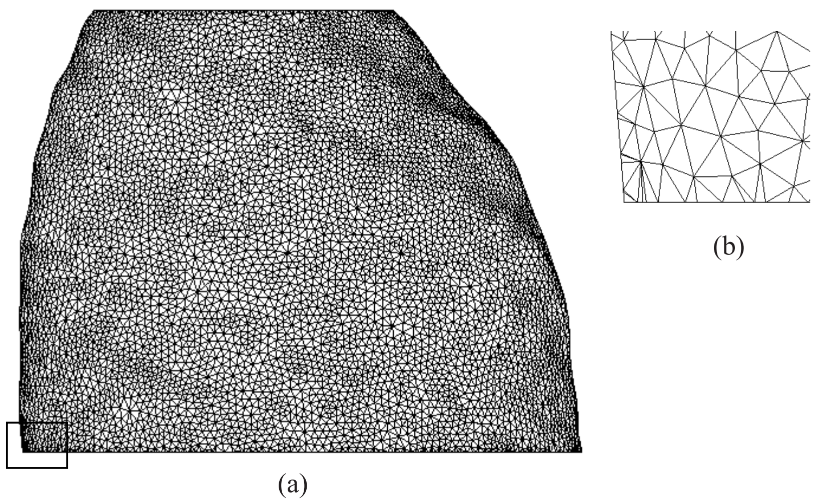


Fig. 7. (a) Surface triangulation over a four-sided patch of the human lung model; (b) Zoomed-in view of the lower left corner.

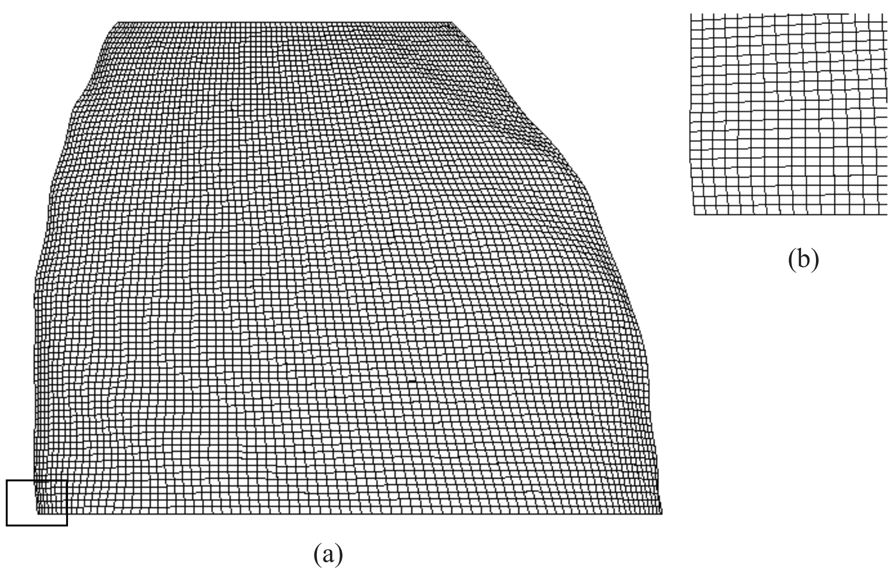


Fig. 8. (a) Structured grid over the four-sided patch of the human lung model obtained using the parametrization; (b) zoomed-in view of the lower left corner.

4.2 Structured Grid

Once a surface triangulation is obtained for the surface patch, a structured surface grid can be obtained by parametrization as explained in Section 2. Figure 8 shows a structured grid over the surface patch of the human lung model obtained by this procedure.

5 Conclusions and Further Work

This paper has shown how Floater's parametrization algorithm can be used to generate structured grids over trimmed patches of NURBS as well as facetted surfaces. In the case of NURBS surfaces, this provides an alternate method of reparametrization to the known method of carpeting. As examples, the paper has used a trimmed singular NURBS surface patch of a volute model and a four sided facetted surface patch of a human lung model. The results obtained on these examples show that this is a good alternative method for structured grid generation. In this paper, only one kind of coordinates λ_{ij} producing shape preserving parametrization has been used. There seems to more scope in the choice of the coordinates λ_{ij} to see if they can be computed differently to produce a parametrization that will result in structured orthogonal grids. More work can be done along this direction.

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