

ERROR ESTIMATES FOR HESSIAN-BASED MESH ADAPTATION ALGORITHMS WITH CONTROL OF ADAPTIVITY

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ABSTRACT

One of the main goals of unstructured mesh adaptation algorithms is to achieve better discretization error with a smaller number of mesh elements. For many problems, the discretization error can be bounded from above by an interpolation error. The main purpose of this paper is to analyze the interpolation error in Hessian-based mesh adaptation algorithms with a control of adaptivity.

Keywords: mesh adaptation, Hessian-based metrics, error estimates

1. INTRODUCTION

One of the main goals of unstructured mesh adaptation algorithms is to achieve better accuracy of a solution with a smaller number of mesh elements. For problems with anisotropic solutions, the optimal adaptive mesh must contain anisotropic elements. One of the interesting results of the approximation theory is that elements with obtuse and acute angles stretched along the direction of minimal second derivative of a solution may be the best elements for minimizing an interpolation error [1, 2]. For many problems, the interpolation error provides an upper bound for a discretization error. The main purpose of this paper is to analyze the interpolation error in metric-based mesh adaptation algorithms.

One way to generate a metric is to use the Hessian (the matrix of second derivatives) recovered from a computed solution. Different methods of Hessian recovery have been studied in [3, 4, 5, 6, 7]. It has been shown in [3] that the adaptive meshes quasi-uniform in the Hessian-based metric result in optimal estimates for the interpolation error.

The optimal adaptive mesh may not be appropriate

for some engineering applications due to strong size-disproportionality of neighboring mesh elements. This disproportionality may increase the LBB constant in mixed finite element discretizations of the Stokes problem [8]. This constant is used to evaluate the difference between the approximation and interpolation errors in the pressure. The more this difference, the less accurate may be the Hessian-based approach for resolving pressure singularities. The solution for some of the problems mentioned above is to modify (for example, to smooth) the Hessian-based metric. A properly chosen modification which preserves the main properties of the Hessian-based metric defines success of many engineering simulations. In this paper, we develop new error estimates which allow us to understand effects of different metric modifications on the interpolation error.

A modified metric allows us to enforce the mesh adaptation in regions of physical interest. Similar objectives are typical for the goal oriented adaptive methods [9]. The main advantage of metric-based methods is a simple control of metric properties and properties of resulting meshes.

The paper contents is as follows. In Section 2, we in-

roduce the notion of a quasi-optimal mesh. In Section 3, we describe a control strategy for mesh generation algorithms based on a metric induced by the Hessian of a discrete solution. In Section 4, we prove the error estimates for the proposed controlled strategy. In Section 5, we consider a numerical example confirming the theoretical estimates.

2. QUASI-OPTIMAL MESHES

Let $\Omega \in \mathbf{R}^3$ be a polyhedral domain and Ω^h be a conformal tetrahedrization of Ω :

$$\Omega^h = \bigcup_{i=1}^{\mathcal{N}(\Omega_h)} e_i,$$

where $\mathcal{N}(\Omega_h)$ denotes the number of mesh elements.

Let $C^k(D)$ be the space of functions over domain $D \subset \bar{\Omega}$ with continuous partial derivatives up to the order k . We shall use notation $P_1(\Omega^h)$ for the space of functions continuous in Ω and linear over each mesh element. Let \mathcal{P}_{Ω^h} be a projector onto this space. An example of such a projector is the linear interpolation operator. The error estimates presented in this paper are given in space L_∞ with the norm

$$\|u\|_{L_\infty(\Omega)} = \sup_{x \in \Omega} |u(x)|.$$

Definition 1 *Let $u \in C^0(\bar{\Omega})$ and \mathcal{P}_{Ω^h} be given. The mesh $\Omega_{opt}^h(N_T, u)$ consisting of at most N_T elements is called optimal if it is a solution of the optimization problem*

$$\Omega_{opt}^h(N_T, u) = \arg \min_{\Omega_h: \mathcal{N}(\Omega_h) \leq N_T} \|u - \mathcal{P}_{\Omega^h} u\|_{L_\infty(\Omega)}. \quad (1)$$

The existence of the optimal mesh has been analyzed in [1, 10, 11] for triangular meshes. In [1, 11], the existence of the optimal triangulation was conditioned by the existence of a curvilinear coordinate transformation resulting in the canonic Hessian. The different analysis based on continuity of the L_∞ -norm with respect to coordinates of mesh nodes has been developed in [10].

The main purpose of this paper is the theoretical analysis of one of the solution strategies for problem (1). This strategy results in a Hessian-based mesh adaptation method. The optimization problem is reformulated as the problem of generating a mesh which is quasi-uniform in a metric space generated by the Hessian of $\mathcal{P}_{\Omega^h} u$.

Let H be the Hessian of function u and H^h be a discrete Hessian recovered from the discrete solution $u^h \equiv \mathcal{P}_{\Omega^h} u$. Since the Hessian H^h is symmetric, the

following spectral decomposition is possible:

$$H^h = W^T \Lambda^h W, \quad \Lambda^h = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

where W is an orthonormal matrix and λ_i , $i = 1, 2, 3$, are eigenvalues of H^h . Using the last formula, we define the following metric:

$$|H^h| = W^T |\Lambda^h| W, \quad |\Lambda^h| = \begin{bmatrix} |\lambda_1| & 0 & 0 \\ 0 & |\lambda_2| & 0 \\ 0 & 0 & |\lambda_3| \end{bmatrix}. \quad (2)$$

Hereafter, we shall assume that the metric $|H^h|$ is non-singular. In practice, zero eigenvalues are replaced by a small positive constant.

There exist several criteria in the literature for checking that a tetrahedron $e \in \Omega^h$ is quasi-uniform in a metric G . In this paper we use the criterion proposed in [3, 5, 12]. Let $|e|_G$ be the volume of tetrahedron e in metric G and $|\partial\partial e|_G$ be the total length of tetrahedron edges in metric G . We define a number $Q(\Omega^h)$ as follows:

$$Q(\Omega^h) = \min_{e \in \Omega^h} Q(e)$$

where

$$Q(e) = 6 \sqrt[4]{2} \frac{|e|_G}{|\partial\partial e|_G^3} F\left(\frac{|\partial\partial e|_G}{6h^*}\right), \quad (3)$$

$$|\Omega^h|_G = \sum_{e \in \Omega^h} |e|_G \quad \text{and} \quad h^* = \sqrt[3]{\frac{12|\Omega^h|_G}{\sqrt{2}N_T}}.$$

It is obvious that $|\Omega^h|_G$ is the volume of the computational domain in metric G and h^* is the average length of a tetrahedron edge. Thus, the number $Q(\Omega^h)$ depends on the metric and the number of mesh elements. The last factor in (3) controls the size of the mesh element. The function $F(x)$ can be an arbitrary continuous function, $0 < F(x) \leq 1$, with the only maximum at point 1, $F(1) = 1$. The second factor controls the shape of the mesh element. The first factor scales the value of $Q(e)$ to interval $[0, 1]$. Thus, the maximal value of $Q(\Omega^h)$ is attained when all mesh elements are equilateral (in metric G) tetrahedra with edge length h^* . Later in this paper, we shall refer to $Q(e)$ and $Q(\Omega^h)$ as the *element quality* and *mesh quality*, respectively. In addition, we shall refer to a mesh quasi-uniform in metric G as the G -quasi-uniform mesh.

For complex geometries, the mesh quality of optimal meshes is usually less than 1. In the process of mesh generation, we require $Q(\Omega^h) \sim Q_0$ where $Q_0 \sim 0.2$. It has been shown in [3, 13] that the resulting meshes are quasi-optimal, i.e. they are approximate solutions of problem (1). In other words, the quasi-optimal meshes still result in the optimal error estimates.

3. CONTROL OF ADAPTIVITY

As it was mentioned in the introduction, the quasi-optimal mesh may not be appropriate for some engineering applications due to strong size-disproportionality of neighboring mesh elements. The solution proposed by many researchers is either to smooth the mesh or to modify the tensor metric.

The second approach is computationally more effective, since the majority of mesh generation algorithms perform better when the metric is smooth. In context of the Hessian-based metric, the adaptation process may be controlled by modifying eigenvalues in spectral decomposition (2). Since the actual size of a mesh element is implicitly controlled via the third factor, $F(\cdot)$, in the element quality definition, we may use local criteria for modifying the metric. In this paper, we analyze theoretically the effect of such a control strategy on the interpolation error. As examples of particular control strategies, we shall consider the following modifications of entries of Λ^h :

$$\tilde{\lambda}_i = \arg \max_{\lambda_1, \lambda_2, \lambda_3} |\lambda_i|, \quad i = 1, 2, 3, \quad (4)$$

and

$$\tilde{\lambda}_i = \lambda_i \omega(x), \quad i = 1, 2, 3, \quad (5)$$

where $\omega(x) > 0$ is a weight function.

The first criterion (4) results in isotropic meshes adapted to the maximal solution curvature. These meshes are similar to hierarchical locally refined meshes whose refinement criterion is based on the equidistribution of the L_∞ -norm of a *posteriori* estimated error. The second criterion allows us to focus the mesh adaptation in regions of physical interest ($\omega > 1$) and/or to relax an undesirable mesh refinement ($\omega < 1$).

Similar objectives are typical for the goal oriented adaptive methods [9]. The main advantage of our approach is its easy implementation in black-box mesh generation methods based on the notion of a metric. The theoretical estimates derived in the next section will allow us to understand effects of different control strategies on the interpolation error.

We denote by \tilde{H}^h the modified Hessian:

$$\tilde{H}^h = W^T \tilde{\Lambda}^h W, \quad \tilde{\Lambda}^h = \begin{bmatrix} \tilde{\lambda}_1 & 0 & 0 \\ 0 & \tilde{\lambda}_2 & 0 \\ 0 & 0 & \tilde{\lambda}_3 \end{bmatrix}, \quad (6)$$

and by $|\tilde{H}^h|$ the associated metric, $|\tilde{H}^h| = W^T |\tilde{\Lambda}^h| W$.

4. ERROR ESTIMATES

The following notations are used in this section. The continuous Hessian of function u is denoted by H . The

discrete Hessian H^h recovered from the discrete solution u^h is assumed to be constant inside each mesh element. We denote by H_Δ its value inside a tetrahedron Δ . Let \tilde{H} be the continuous modified Hessian and \tilde{H}^h defined in (6) be its piecewise constant approximation. We denote by \tilde{H}_Δ the value of \tilde{H}^h inside a tetrahedron Δ . Note that in practice the discrete Hessian is piecewise linear and continuous rather than piecewise constant. We consider here the latter case for the sake of simplicity of the presentation. Finally, we shall use notation $C(z)$, $C_1(z)$ and $C_2(z)$ for generic constants depending on parameter z and independent of other parameters.

The following spectral decompositions hold:

$$H_\Delta = W_\Delta^T \Lambda_\Delta W_\Delta \quad \text{and} \quad \tilde{H}_\Delta = W_\Delta^T \tilde{\Lambda}_\Delta W_\Delta$$

where Λ_Δ and $\tilde{\Lambda}_\Delta$ are diagonal matrices. Similar decompositions can be written for continuous metrics H and \tilde{H} . Let $\Lambda(x)$ and $\tilde{\Lambda}(x)$ be the corresponding eigenmatrices. We assume that the eigenvalues are ordered in such a way that $|\lambda_1(\Lambda)| \leq |\lambda_2(\Lambda)| \leq |\lambda_3(\Lambda)|$. Let $\rho(\Lambda)$ denote the spectral radius of Λ and $\text{Cond}(\Lambda)$ be the condition number of Λ .

The following two lemmas proved in [3] play the important role in our analysis.

Lemma 1 *Let Δ be a tetrahedron and $u_2 \in P_2(\Delta)$ be a quadratic function with the nonsingular Hessian H_2 such that $H_2 = W_2^T \Lambda_2 W_2$. Then*

$$C|\hat{\Delta}|^{\frac{2}{3}} \leq \|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)} \leq \hat{r}^2/2, \quad (7)$$

where \hat{r} is the circumradius of tetrahedron $\hat{\Delta}$ which is the image of Δ under the transformation $\hat{x} = R(x)$, $R = \sqrt{|\Lambda_2|} W_2$, reducing H_2 to the canonical form

$$\hat{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

Lemma 2 *Let G^1, G^2 be two constant metrics defined on a tetrahedron Δ such that,*

$$|G_{ps}^1 - G_{ps}^2| \leq \varepsilon, \quad p, s = 1, 2, 3.$$

Let $Q_{G^1}(\Delta)$ and $Q_{G^2}(\Delta)$ be the tetrahedron qualities in metrics G^1 and G^2 , respectively, and

$$Q_{G^1}(\Delta) \geq Q_0,$$

Then, for a sufficiently small $\varepsilon > 0$,

$$Q_{G^2}(\Delta) \geq Q_0 \cdot (1 - C\varepsilon/\lambda_1(G^1))^5.$$

Let Ω_{opt}^h be a solution of the optimization problem (1). The following theorem holds.

Theorem 1 Let $u \in C^2(\bar{\Omega})$, the Hessians H and \tilde{H} be nonsingular in Ω , and \mathcal{P}_{Ω^h} be the interpolation operator. Let Ω^h be the $|\tilde{H}^h|$ -quasi-uniform mesh such that $Q(\Omega^h) > Q_0$. Furthermore, let for any tetrahedron $\Delta \in \Omega_{opt}^h$ and the tetrahedron $\Delta^* \in \Omega^h$ where $\|u - \mathcal{P}_{\Omega^h} u\|_{L^\infty(\Omega)}$ is attained the following estimates hold:

$$\begin{aligned} \|H_{ps} - (H_\Delta)_{ps}\|_{L^\infty(\Delta)} &< q_\Delta \lambda_1(|H_\Delta|), \\ \|\tilde{H}_{ps} - (\tilde{H}_\Delta)_{ps}\|_{L^\infty(\Delta)} &< q_\Delta \lambda_1(|\tilde{H}_\Delta|), \end{aligned} \quad (8)$$

for $0 < q_\Delta \leq q < 1$ and $p, s = 1, 2, 3$. If H_{Δ^*} is an indefinite tensor, we assume additionally that

$$\text{Cond}(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) q_{\Delta^*} \leq q. \quad (9)$$

Then

$$\|u - \mathcal{P}_{\Omega^h} u\|_{L^\infty(\Omega)} \leq \alpha(H, \tilde{H}) \|u - \mathcal{P}_{\Omega_{opt}^h} u\|_{L^\infty(\Omega)} \quad (10)$$

where

$$\alpha(H, \tilde{H}) = C(Q_0, q) \max_{x \in \Delta^*} \rho(\Lambda \tilde{\Lambda}^{-1}) \left(\frac{|\Omega|_{|\tilde{H}|}}{|\Omega|_{|H|}} \right)^{2/3}.$$

Conditions (8) may be treated as requirements of relatively small variations of Hessians H and \tilde{H} on any tetrahedron of the optimal mesh Ω_{opt}^h and the worst tetrahedron of the $|\tilde{H}^h|$ -quasi-uniform mesh Ω^h . For instance, the conditions are satisfied for sufficiently refined meshes and functions u such that $\lambda_1(|H|) \geq C$. It is pertinent to note that the last inequality holds for a wide set of anisotropic functions.

Note that $\text{Cond}(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) = |\lambda_3/\lambda_1|$ and $\text{Cond}(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) = 1$ for control strategies given by (4) and (5), respectively. We think that condition (9) can be dropped out using a more sophisticated analysis than that presented below.

Proof. We begin by proving (10) for a piecewise quadratic function u_2 defined by:

$$u_2(x) = \frac{1}{2} (H_\Delta x, x) + (b, x) + d \quad (11)$$

where $b \in \mathfrak{R}^3$ and d are such that

$$\mathcal{P}_\Delta u = \mathcal{P}_\Delta u_2. \quad (12)$$

Here \mathcal{P}_Δ is the restriction of \mathcal{P}_{Ω^h} to tetrahedron Δ . The last equation is equivalent to the system of four linear equations at tetrahedron vertices:

$$u_2(a_i) = u(a_i), \quad i = 1, 2, 3, 4.$$

Since this system is non-singular, (12) has a unique solution.

Let us estimate the circumradius \hat{r} for tetrahedron $\hat{\Delta}$ which is the image of $\Delta^* \in \Omega^h$ under the transformation described in Lemma 1. The spectral decomposition implies that the transformation R from Lemma 1 can be rewritten as follows:

$$R = \Lambda_{\Delta^*}^{1/2} W_{\Delta^*} = \Lambda_{\Delta^*}^{1/2} \tilde{\Lambda}_{\Delta^*}^{-1/2} \tilde{R}, \quad \tilde{R} = \tilde{\Lambda}_{\Delta^*}^{1/2} W_{\Delta^*},$$

where \tilde{R} reduces \tilde{H}_{Δ^*} to the canonical form. Let \tilde{r} be the circumradius of $\tilde{\Delta} \equiv \tilde{R}(\Delta^*)$. Note that

$$Q(\Delta^*) \geq Q_0.$$

The definition of the element quality implies that

$$6\sqrt[4]{2} \frac{|\Delta^*|_{|\tilde{H}_{\Delta^*}|}}{|\partial\partial\Delta^*|_{|\tilde{H}_{\Delta^*}|}^3} \geq Q_0, \quad F \left(\frac{|\partial\partial\Delta^*|_{|\tilde{H}_{\Delta^*}|}}{6h^*} \right) \geq Q_0.$$

The first inequality implies that the tetrahedron $\tilde{\Delta}$ is close to the equilateral one. The definition of function F implies that the tetrahedron perimeter $|\partial\partial\tilde{\Delta}|$ is bounded from above and below by $6h^*$. Thus, we get the following estimate for circumradius \tilde{r} :

$$C_1(Q_0, q) h^* \leq \tilde{r} \leq C_2(Q_0, q) h^*.$$

Note that $\hat{\Delta}$ is obtained by stretching of tetrahedron $\tilde{\Delta}$. Since the tetrahedron $\tilde{\Delta}$ is quasi-equilateral, the circumradius \hat{r} of $\hat{\Delta}$ is bounded from above:

$$\hat{r} \leq C \rho(\Lambda_{\Delta^*}^{1/2} \tilde{\Lambda}_{\Delta^*}^{-1/2}) \tilde{r}.$$

Now, Lemma 1 gives the following estimate:

$$\|u_2 - \mathcal{P}_{\Delta^*} u_2\|_{L^\infty(\Delta^*)} \leq \hat{r}^2/2 \leq C \rho(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) \tilde{r}^2.$$

The estimate for the circumradius \tilde{r} , definition of h^* and inequalities (8) give the following upper bound:

$$\|u_2 - \mathcal{P}_{\Delta^*} u_2\|_{L^\infty(\Delta^*)} \leq \alpha(H, \tilde{H}) \left(\frac{|\Omega|_{|H|}}{N_T} \right)^{\frac{2}{3}}. \quad (13)$$

Let Δ be an arbitrary tetrahedron from the optimal triangulation Ω_{opt}^h . Applying Lemma 1, we get

$$\max_{x \in \Delta} |u_2(x) - \mathcal{P}_\Delta u_2(x)| \geq C |\Delta|_{|H_\Delta|}^{\frac{2}{3}}.$$

Due to (8) and Lemma 2, there exists a constant $C(q)$ such that

$$|\Delta|_{|H_\Delta|} \geq C(q) |\Delta|_{|H|}.$$

Recall that mesh Ω_{opt}^h is quasi-uniform in metric $|H|$. Therefore,

$$\begin{aligned} \|u_2 - \mathcal{P}_{\Omega_{opt}^h} u_2\|_{L^\infty(\Omega)} &\geq C(q) \max_{\Delta \in \Omega_{opt}^h} |\Delta|_{|H|}^{\frac{2}{3}} \\ &\geq C(q) \left(\frac{|\Omega|_{|H|}}{N_T} \right)^{\frac{2}{3}}. \end{aligned} \quad (14)$$

Combining (13) and (14), we prove the statement of the theorem for the quadratic function u_2 . In order to

complete the proof of the theorem, we have to show that the quadratic function u_2 satisfies the following inequalities:

$$C_1(q)\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)} \leq \|u - \mathcal{P}_\Delta u\|_{L_\infty(\Delta)} \quad (15)$$

and

$$\|u - \mathcal{P}_\Delta u\|_{L_\infty(\Delta)} \leq C_2(q)\|u_2 - \mathcal{P}_\Delta u_2\|_{L_\infty(\Delta)}. \quad (16)$$

In order to verify (15) and (16), we take advantage of the multipoint Taylor formula [14, 15]. Let

$$\begin{aligned} E &= u(x) - \mathcal{P}_\Delta u(x) \\ &= \frac{1}{2} \sum_{i=1}^4 (H(c_i)(x - a_i), (x - a_i)) \phi_i, \\ E_2 &= u_2(x) - \mathcal{P}_\Delta u_2(x) \\ &= \frac{1}{2} \sum_{i=1}^4 (H_\Delta(x - a_i), (x - a_i)) \phi_i \end{aligned}$$

where ϕ_i is the P_1 basis function at vertex a_i and $c_i = c_i(x, a_i)$ is some point in Δ . Let us define

$$E_{2,mod} = \frac{1}{2} \sum_{i=1}^4 (|H_\Delta|(x - a_i), (x - a_i)) \phi_i.$$

The first assumption in (8) implies that

$$\begin{aligned} |E - E_2| &\leq \frac{q\Delta\sqrt{3}}{2} |\lambda_1(H_\Delta)| \sum_{i=1}^4 ((x - a_i), (x - a_i)) \phi_i \\ &\leq \frac{q\Delta\sqrt{3}}{2} \sum_{i=1}^4 (|H_\Delta|(x - a_i), (x - a_i)) \phi_i \\ &\leq q\Delta\sqrt{3} \|E_{2,mod}\|_{L_\infty(\Delta)}. \end{aligned}$$

Recall that the transformations reducing H_Δ and $|H_\Delta|$ to canonical forms are identical. Since the image of $\Delta \in \Omega_{opt}^h$ under the above transformation is a quasi-equilateral tetrahedron, the lower estimate in Lemma 1 can be replaced by $C\hat{r}^2$. Thus, we have

$$\|E_{2,mod}\|_{L_\infty(\Delta)} \leq C\|E_2\|_{L_\infty(\Delta)} \quad (17)$$

for both definite and indefinite tensors H_Δ where C is a constant independent of Δ .

The case of $\Delta^* \in \Omega^h$ is more difficult since the image of Δ^* is no longer a quasi-equilateral tetrahedron. Applying Lemma 1, we get

$$\begin{aligned} \|E_{2,mod}\|_{L_\infty(\Omega)} &\leq \hat{r}^2/2 \\ &\leq C \frac{\hat{r}^2}{|\tilde{\Delta}|^{2/3}} \|E_2\|_{L_\infty(\Omega)} \\ &\leq C \text{Cond}(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) \|E_2\|_{L_\infty(\Omega)}. \end{aligned}$$

Now, the theorem assumptions (8) and (9) imply

$$\|E\|_{L_\infty(\Delta)} \leq (1 + Cq\sqrt{3}) \|E_2\|_{L_\infty(\Delta)}.$$

For sufficiently small q , we get the lower estimate:

$$(1 - Cq\sqrt{3}) \|E_2\|_{L_\infty(\Delta)} \leq \|E\|_{L_\infty(\Delta)}.$$

This proves inequalities (15), (16) and the statement of the theorem. \square

The important consequence of Theorem 1 is the optimal error estimate for the control strategy defined by (5). Let $L_\infty^\omega(\Omega)$ be the weighted space with the norm

$$\|u\|_{L_\infty^\omega(\Omega)} = \sup_{x \in \Omega} |\omega(x)u(x)|,$$

where $\omega(x) > 0$ is a weight function.

Corollary 1 *Let $\tilde{H}_\Delta = \omega_\Delta H_\Delta$ and $\omega \leq 1$. Then, the following optimal estimate holds:*

$$\|u - \mathcal{P}_{\Omega^h} u\|_{L_\infty^\omega(\Omega)} \leq C(Q_0, q) \|u - \mathcal{P}_{\Omega_{opt}^h} u\|_{L_\infty(\Omega)}.$$

Proof. Note that $\tilde{\Lambda}_\Delta = \omega_\Delta \Lambda_\Delta$. Then, for the worst tetrahedron $\Delta^* \in \Omega^h$, we use (16) to get the following chain of estimates:

$$\begin{aligned} \|u - \mathcal{P}_{\Delta^*} u\|_{L_\infty^\omega(\Delta^*)} &\leq \omega_{\Delta^*} \|u - \mathcal{P}_{\Delta^*} u\|_{L_\infty(\Delta^*)} \\ &\leq C_2(q) \omega_{\Delta^*} \|u_2 - \mathcal{P}_{\Delta^*} u_2\|_{L_\infty(\Delta^*)} \\ &\leq C_2(q) \omega_{\Delta^*} \rho(\Lambda_{\Delta^*} \tilde{\Lambda}_{\Delta^*}^{-1}) \tilde{r}^2 \\ &= C_2(q) \tilde{r}^2, \end{aligned}$$

The statement of the corollary follows from the estimate on the circumradius \tilde{r} , the definition of $|\Omega|_{|\tilde{H}|}$,

$$|\Omega|_{|\tilde{H}|} = \int_{\Omega} \sqrt{\det(|\tilde{H}|)} \leq \int_{\Omega} \sqrt{\det(|H|)} = |\Omega|_{|H|}.$$

and estimates (14), (15). \square

5. NUMERICAL EXAMPLE

In this section, we consider the test example from [16, 17] and explain the results of numerical experiments from the theoretical viewpoint. Let u be the solution of the homogeneous Dirichlet boundary value problem for the Poisson equation in the domain Ω with one reentrant corner, $\Omega = (0, 1)^3 \setminus [0, 0.5]^3$:

$$-\Delta u = f$$

where $f(x)$ is a singular right hand side, $f(x) = 1/|x - x_0|$, and $x_0 = (0.5, 0.5, 0.5)$. Properties of solution u are investigated in [18]. The solution possesses weak anisotropic edge singularities and a strong point singularity at the reentrant corner x_0 due to the singular right hand side.

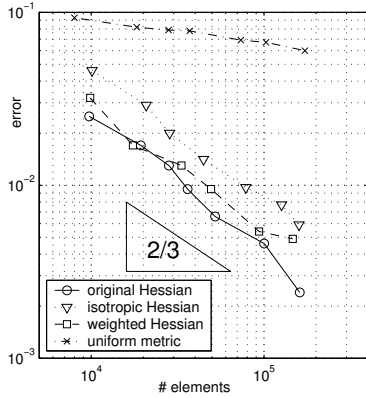


Figure 1: Discretization errors for four metrics.

Let us consider three different metrics. The first metric is generated by the Hessian recovered from the discrete solution. The second and the third metrics are obtained from the first one by using control strategies (4) and (5), respectively. Let the weight function be $\omega(x) = 1/f(x)$. Since $f(x)$ is the sum of second derivatives of $u(x)$, the weight function enforces mesh adaptation along domain edges rather than in vicinity of the point singularity at x_0 . Therefore, we measure the error in domain $D = \Omega \setminus \bar{B}$ which does not contain the ball B of radius $r_0 = 0.1$ centered at x_0 .

As shown in Fig. 1, the errors for all three metrics are proportional to $\mathcal{N}(\Omega_h)^{-2/3}$ which is the asymptotically optimal result. The error on quasi-uniform meshes generated by the uniform metric, is not optimal. The trace of the quasi-optimal mesh is shown in Fig. 2. For the second (isotropic) metric, the worst tetrahedron Δ^* in the quasi-optimal mesh is located close to one of the three reentering edges (see Fig. 3). Since the edge anisotropy is weak, the factor $\alpha(H, \bar{H})$ from Theorem 1 is relatively small. The weighted Hessian preserves main properties of the original Hessian everywhere in the computational domain except a neighborhood of the point singularity (see Fig. 4). Therefore, the factor $\alpha(H, \bar{H})$ is smaller than in the previous case. Hence, the error is closer to the error for the original Hessian-based metric.

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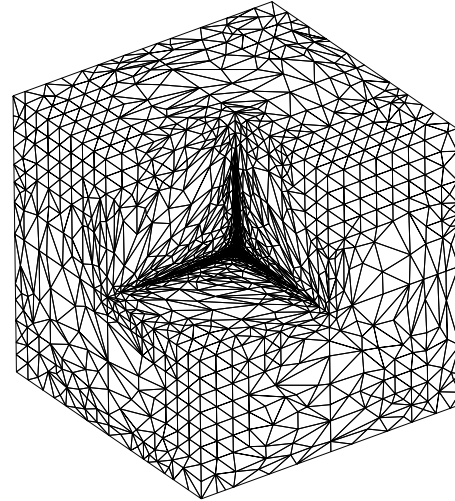


Figure 2: The trace of the quasi-optimal mesh.

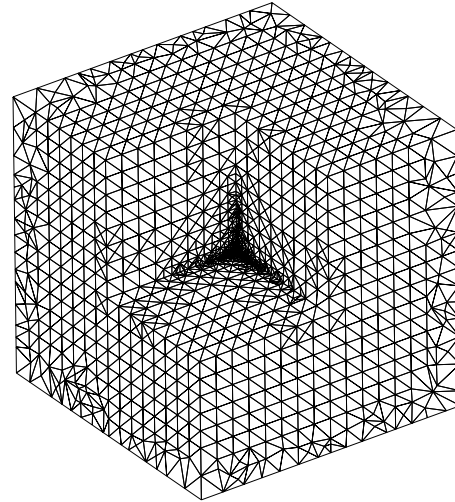


Figure 3: The trace of the isotropic mesh.

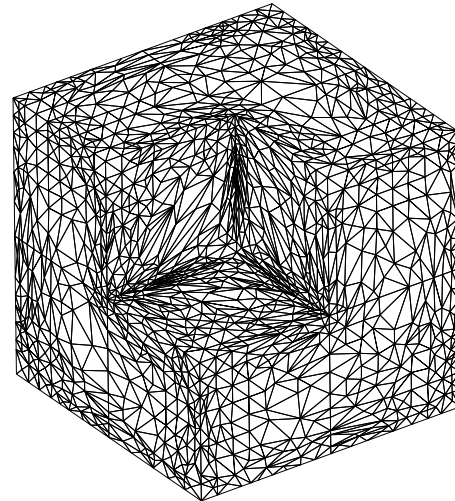


Figure 4: The trace of the weighted mesh.

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