TOPOLOGY MODIFICATION OF HEXAHEDRAL MESHES USING ATOMIC DUAL-BASED OPERATIONS

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ABSTRACT

Topology modification of hexahedral meshes has been considered difficult due to the propagation of topological modifications non-locally. We address this problem by working in the dual of a hexahedral mesh. We prove several relatively simple combinatorial aspects of hex mesh duals, namely that they are both complexes of simple polytopes as well as simple arrangements of pseudo-hyperplanes. We describe a set of four atomic dual-based hex topology modifications, from which the flipping operations of Bern et. al can be constructed. We also observe several intriguing arrangements and modification operations, which we intend to explore further in the future.

Keywords: hexahedral mesh generation; arrangements; dual; simple polytope.

1. INTRODUCTION

Many finite element analysis practitioners prefer using all-hexahedral meshes. These meshes are believed to yield more accurate solutions for a given computational expense, especially in the non-linear analysis regime (theoretical studies investigating this issue are appearing now [1], joining the more empirical studies on the subject [2][3]).
Generating all-hexahedral meshes suitable for finite element analysis remains an active area of research[4]. Working with hexahedral meshes has been considered difficult in part because of the non-local nature of connectivity modifications inside hex meshes. That is, until recently there were relatively few known options for modifying the topology interior to a hexahedral mesh which did not propagate through the mesh. This is in contrast to tetrahedral meshes, where local connectivity modifications are a crucial part of most robust tetrahedral mesh generation algorithms [5]. Furthermore, local connectivity modifications have also played an important role in all-quadrilateral meshing [6][7]. It is reasonable to expect that post-meshing topology modifications will play an important role in any successful automatic hexahedral meshing algorithm. In addition to mesh generation, local connectivity modification in hexahedral meshes could also be useful for adaptive refinement and for local mesh quality improvement, analogous to its counterpart in tetrahedral meshes. For all these reasons, we believe local connectivity modification in hexahedral meshes to be an important technology.

1.1. Hexahedral Dual

We study hex topology modification in the dual, for reasons which will become apparent later in this paper. The dual of a hex mesh is analogous locally to Voronoi diagrams for tetrahedral meshes, where (in three dimensions) each primal entity of dimension *k* (e.g. node, edge, face) has a corresponding *(3-k)-cell* (e.g. 3-cell, 2-cell, 1-cell). Non-locally, the dual of a hex mesh has special properties: 1-cells and 2-cells can be grouped into larger structures which have non-local extent in the mesh. Indeed, the dual of a hexahedral mesh can be viewed as an arrangement of surfaces, with (3-k)-faces in the arrangement corresponding to k-dimensional entities in the primal mesh. Although this characteristic was recognized quite some time ago[8], its application to hexahedral meshes was not recognized until much later[9]. The physical interpretation of these surfaces is of topologically 2d layers of hexahedral elements; pairwise intersections of these surfaces represent columns of hex elements in the mesh. Both these structures are typically non-local in the mesh, and are sometimes self-intersecting. These structures are referred to as sheets and chords, respectively, and have been used in the development of the Whisker Weaving dual-based hex meshing algorithms[11][12].
One useful way of representing dual sheets (from here on referred to as sheets) is by viewing their 2d projection, where

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intersections with other sheets are depicted by lines (pairwise intersection) and vertices (three-way intersection). These "sheet diagrams" simplify the study of the dual arrangement by representing it as a series of 2d projections[11].
An example of a simple hex mesh, its dual, and its representation as a series of sheet diagrams is shown in (fig). The outer loop of each sheet represents the sheet's intersection with the outer boundary of the mesh. The lines of intersection with other sheets we refer to as chords; on the sheet diagrams intersections with other chords represent hex elements, and intersections with the outer loop represent the emergence of a chord at a face on the boundary of the mesh. Loop-intersecting and interior vertices on sheet diagrams are sometimes labeled with the face/hex to which they correspond, respectively.

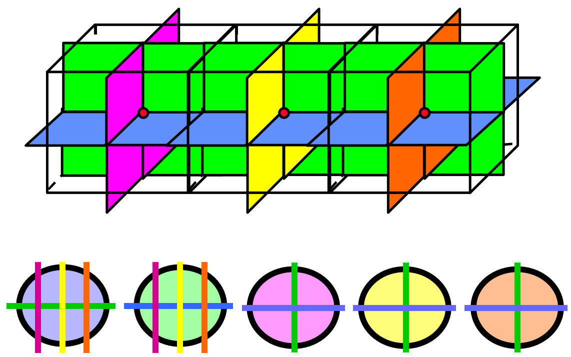

Figure 1: Hex mesh with three elements (top); dual surfaces (sheets) and dual vertices shown. Two-dimensional projection into "sheet diagrams" shown below.

One can see that:

- Each chord appears on the two sheets whose intersection form the chord
- A hex element appears as a vertex on three sheet diagrams as the pairwise intersection of three chords
- Each 1-cell (or segment of a chord) and 2-cell in the sheet diagrams represents a face and edge in the primal mesh, respectively

Sheet diagrams, and the 1-cells and 2-cells on them, are used later in this paper to depict hex topology modifications.

1.2. Prior Work

Relatively little work has been done in the area of hex mesh topology modification.
This subject was treated peripherally in the development of Whisker Weaving [13], and in the study of "knife" element resolution [14]. While some of the structures being used in the current work were evident, no effort was made to utilize them in local mesh improvement aside from the constructive process.
Knupp & Mitchell studied hex mesh topology optimization in connection with hex meshes arising from tetrahedral mesh subdivision [15]. The topology modification operations described there were similar to some described in this paper, but the completeness of these operations and their unification under a common theoretical framework was not addressed.
Bern et. al research topology modifications of hex meshes in connection with "flips" of hex meshes, where it is shown that there exists a complete set of hex mesh transitions, each of whose outer boundary corresponds to the six faces of a cube [17]. More recently, this research is being extended to consider face collapsing inside the mesh [18]. This work has played an important role in our research, both as inspiration for starting the work as well as serving as a benchmark in various ways. We describe these links later in this paper.
As mentioned before, the general approach of using local mesh improvement in a post-meshing context has been used in both the tetrahedral [5]and quadrilateral [6][7] meshing areas. While we believe the various cleanup methods demonstrated for quads could be unified under a common framework similar to the current work, no attempt has been made to do that since this seems to be treated adequately by prior methods.

1.3. Current Work

In this paper we outline the basic operations of and motivations for locally modifying the topology of an existing hexahedral mesh. Our initial goal has been to define a complete set of local, atomic operations on the topology of a hexahedral mesh. By "complete" we mean that we can transform a given hexahedral mesh to any other with the same quadrilateral boundary using only these operations. By "local" we mean that the connectivity altered by the sequence of operations is local to the actual hexes targeted by the operations. This boundary is denoted in the following sections by a dotted line which replaces the outer boundary of a sheet diagram, i.e. it is the group of quadrilateral faces bounding the region of interest, just as the outer boundary of a normal sheet diagram is a group of quadrilaterals on the outer boundary of the solid. Finally, the use of "atomic" above is intended to indicate that the operations we seek to define are in some sense irreducible, that is, cannot be done as a sequence of other atomic operations. In particular, we show that the flipping operations presented in [18] can be reproduced as a series of these elementary operations.
The contribution of this work is the reduction of other known hex mesh topology modification operations into a much smaller set of atomic operations. This work also unifies these operations under a common theme of modifications to a simple arrangement of surfaces, while showing how these operations can be directed toward specific mesh improvement or mesh modification for other purposes.
The remainder of this paper is arranged as follows. Section 2 reviews elements of the dual of a hexahedral mesh, and proves several new results for those types of duals. Section 3 describes the general approach we take to hex mesh topology modification. Section 4 describes the atomic operations we use to describe others' topology modification operations. Section 5 relates these operations to the flipping operations described by Bern et. al. Section 6 gives conclusions and future directions for this work.

2. THE HEXAHEDRAL MESH DUAL

Before describing our work in hex topology modification, we describe here some characteristics of the dual in terms of complexes of polytopes and arrangements. Describing the hex dual in these terms allows us to take advantage of the wealth of prior work on combinatorial relations for polytope complexes and arrangements (e.g. [8]).

To define the dual, we start by identifying the hexahedral mesh as a cell complex P whose cells P_i are hexahedral. Then, the dual is defined as a one-to-one mapping $\Psi(P)$: $P \rightarrow P^*$ which preserves but reverses incidence relations:

$$P_i \subset P_j \leftrightarrow \Psi(P_j) \subset \Psi(P_i)$$

The dual P^* we identify as a polytope complex, with each k-face dual to a $(d\text{-}k)$-face in the hexahedral mesh.

A great deal of results have been published on various combinatorial characteristics of polytope complexes and arrangements. In particular, if it can be shown that the polytope or complex being considered is simple[2], a richer set of combinatorial relations is available. Therefore, we first address these issues for hexahedral mesh duals.

2.1. The hex dual is a complex of simple polytopes

Theorem 1: The dual of a hexahedral mesh is a complex of simple polytopes.

Proof: Consider a hex element P, one of its bounding vertices $V(P)$, and the sets of 1-faces $F^1(P,V)$ and 2-faces $F^2(P,V)$ of P incident on V. Because a hexahedron is a simple polytope, $card(F^2(P,F_i^0)) = d$. Also, since every k-face of a simple polytope is formed by the intersection of $(d\text{-}k)$ supporting faces of the polytope, $card(F^1(P,F_i^0)) = \binom{d}{d-1} = d$. Since the dual preserves incidence relations (and does not change dimension),

$$\Psi(P) \subset \Psi(F^2(P,V)) \subset \Psi(F^1(P,V)) \subset \Psi(V),$$

or

$$V^*(P^*) \subset F^{1*}(V^*,P^*)) \subset F^{2*}(V^*,P^*)) \subset P^*,$$

with

$$card(F^{2*}(V^*,P^*)) = card(F^1(P,V) = d \text{ and}$$

$$card(F^{1*}(V^*,P^*)) = card(F^2(P,V) = d. \qquad (1)$$

However, because the dual relation between entities is one-to-one and we chose an arbitrary vertex and any of the hex elements bounding it, (1) must apply to all the dual cells as well as all the dual vertices bounding those cells. Therefore, the dual cells must be simple polytopes.

□

[2] A simple polytope is one whose k-faces are formed by the intersection of $(d\text{-}k)$ facets of the polytope. For example, for a 3-dimensional polytope, each 1-face (edge) is formed by the intersection of 3-1=2 facets (faces).

2.2. The hex dual is a simple arrangement of hyperplanes

Theorem 2: The dual of a hexahedral mesh forms a simple arrangement of hyperplanes.

Proof: We start by deriving some incidence relations for vertices in a simple arrangement, by constructing a simple arrangement containing a single vertex. Consider a single surface, h_a, passing through a point p. Now, add a second surface, h_b,different from the first, also passing through p. The intersection of the two planes is a line contained in each plane, which partitions each plane into two facets. Similarly, add a third plane, h_c, which also passes through p, and is affinely independent of the other two planes (so its intersection with each plane generates a new line on that plane). Note that the third plane has to be independent of the first two, since for simple arrangements each *1*-face can only be formed by the intersection of d-*1*=2 surfaces. When h_c is added, a new line of intersection is formed with each of the other planes, such that each plane is partitioned by two lines, formed by its intersection with the other two planes. Since all the lines pass through p, each plane is partitioned into 4 *2*-faces. The vertex at p, formed by the intersection of these three hyperplanes, is therefore connected to 12 *2*-faces of the arrangement. The number of *1*-faces incident on the vertex is computed by recognizing that three lines pass through, and are each partitioned in half by, p. Therefore, 6 *1*-faces are incident on the vertex. Finally, because each plane is affinely independent of the others, and passes through p, they each partition all existing cells in half (where R^3 is taken as a single cell), so the vertex is connected to 2^3 = 8 *3*-faces. No more planes can be added which pass through p, by definition of a simple arrangement. Therefore, for a simple arrangement:

$$f_1(V) = 6 \qquad (2)$$
$$f_2(V) = 12$$
$$f_3(V) = 8$$

Next, we consider a single hexahedral element P in the mesh. A hex is a cuboid polytope, which by definition is bounded by $2d$ $(d\text{-}1)$-faces (e.g. one pair for each parametric direction):

$$f_{d-1}(P) = 2d$$

Again, because P is a simple polytope, it obeys the Dehn-Sommerville relations, and specifically for d=3, we have[19]:

$$f_0(P) = 2f_2 - 4$$
$$f_1(P) = 3f_2 - 6$$

Combining these, we get

$$f_0(P) = 4(d-1) = 8 \qquad (2)$$
$$f_1(P) = 6(d-1) = 12$$
$$f_{d-1}(P) = 2d = 6$$

Now, we apply the dual transformation to these relations. As in the previous section, d is unchanged, and each k-face corresponds to a $(d\text{-}k)$-face in the dual. Since Eqs. (2) apply to P in the primal, they apply to $\Psi(P) = V^*$ in the dual; furthermore, the cardinalities don't change, since Ψ preserves incidence relations. Therefore, Eqs. (2) become:

$$f^*_3(V) = 8$$
$$f^*_2(V) = 12 \qquad (3)$$
$$f^*_1(V) = 6$$

However, these are the same numbers of incidences as occur in simple arrangements. No other *1*-, *2*- or *3*-faces can be connected to V^*, because by dual correspondence that would mean adding extra *2-*, *1-* or *0*-faces to the original polytope, respectively, in which case the polytope would no longer be a hexahedron. Since all vertices in the dual arrangement of a hex mesh must have incidences as in (3), and since all simple *d*-dimensional arrangements must have vertices with valences as in (2), then the dual of a hex mesh must form a simple arrangement.

□

Note: *Most combinatorial relations for arrangements assume that hyper-surfaces intersect each other at most a constant number of times s. However, Theorems 1-2 above use only information local to the polytope/vertex combinations. Therefore, as long as the 2-faces local to the polytope and vertex are distinct and not formed from the same hyper-surface locally, we argue that then the Dehn-Sommerville relations hold locally and multiply-intersected hyper-surfaces do not change the results.*

2.3. General comments

By themselves, Theorems 1-2 do not seem to have much relevance to the subject of hexahedral meshing. However, showing that a hexahedral mesh dual is a simple arrangement allows us to take advantage of the Dehn-Sommerville relations, which govern the counting relations between entities in any valid hex mesh. We plan to use these relations to define permissible states, and transitions between those states, in the dual arrangement, thereby proving the completeness of our set of atomic operations. This paper is a work in progress toward that goal.

3. GENERAL APPROACH

One way to look at the hex mesh dual is that hex elements are induced by the intersection of 3 dual sheets pairwise. It follows that mesh connectivity can be modified by locally "deforming" sheets to produce more intersections with other sheets in the neighborhood of the deformation[3]. If only sheet interiors are deformed, mesh connectivity outside the deformed region is unchanged. Thus, the topology modification is local.

We characterize local hex mesh topology modification in terms of atomic, local combinatorial modifications to the dual arrangement, and show that these operations can describe other known topology modifications in hexahedral meshes. Atomic modifications to the arrangement are defined as the smallest units of combinatorial change to the arrangement which keeps the arrangement simple. Using these atomic operations, non-local topology modifications can be accomplished by applying the operations sequentially. Since each operation is reversible, this sequence includes some operations applied in the forward sense and others in the reverse sense. Although some operations introduce what would normally be considered poor quality elements in the primal mesh, this is an intermediate state which gets removed eventually by other operations. This is analogous to the construction of quadrilateral meshes by various algorithms, where poor-quality quads are formed initially but then removed a-posteriori to meshing.

There are two primary applications of these modification operations: quality improvement, and adaptive refinement/coarsening, analogous to the application of analogous operations in tetrahedral meshes. (In practice, these applications are not completely independent of each other.) We speculate that applying our dual-based operations will require "steering" sheet deformations toward areas of interest, either to eliminate structures in the arrangement which induce poor quality, or to enrich/coarsen the arrangement in areas where adaptive refinement or coarsening is desired. We leave this subject for future work.

4. ATOMIC OPERATIONS

We present candidates for atomic, local, dual-based hex topology modification operations in turn in subsequent sections. Formally, we identify four reversible operations: chord push, hex push, minimal pillowing, and face/ring collapse. The first two operations modify interior regions on three and four sheets, respectively. Minimal pillowing creates a new sheet interior to the mesh along with modifying two existing sheets. Face collapse merges two sheets by forming and joining a new interior boundary on each sheet. We also describe two higher-level operations, the triple-chord push and triple-hex push, which are combinations of elementary operations which occur frequently when describing topology modifications studied in previous works.

4.1. Chord Push

The minimum combinatorial change one can make in an arrangement is the addition or removal of *d*-facets in the arrangement. Since we restrict ourselves to locally-simple arrangements, new facets must be introduced in a way which preserves that characteristic. That means, for example, that the arrangement cannot be modified by deforming two dual surfaces such that they meet only at a single vertex or 0-facet. By this reasoning, then, the smallest (constructive) combinatorial change in a simple arrangement is to deform two sheets along a common third sheet which intersects both, such that they intersect locally. This "chord push" operation is depicted in Figure 2; starting with the arrangement shown on top, the 2-cells **ab** and **cd** are deformed such that they intersect, modifying the arrangement to look like Figure 2, bottom. A chord push operation introduces two new 3-cells, 4 new 2-cells, six new 1-cells, and two new 0-cells in the dual. In the primal, two new hex elements and nodes are introduced.

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3 Since we use the dual primarily to study mesh topology (i.e. we ignore the issue of geometric embedding of the dual sheets), sheet "deformations" are discrete in nature; that is, a sheet deformation is only meaningful if it modifies the combinatorial properties of the dual arrangement.

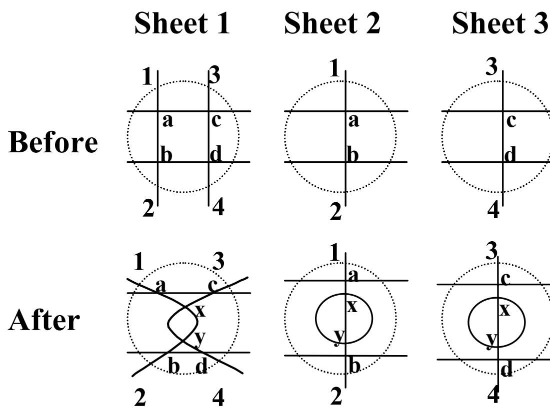

Figure 2: The chord push operation in the dual. Intersecting two 1-cells (ab and cd) to form 2 new 0-cells (x and y), 6 new 1-cells (ax, cx, xy (2), yb, yd), 6 new 2-cells (axc, bdy, xyx (4)), and 2 new 3-cells. New xy 1-cells make up blind chord on sheets 2 and 3; xy 1-cells on 12 and 34 chords correspond to 1-cells ab and cd, resp., in original arrangement (top, sheets 2 and 3).

Since there are no combinatorial changes outside the bounding circle on each sheet, the chord push operation is local.

In the primal, the interpretation of a chord push is an opening of two interior faces, creating two deformed hexes that share four faces (Figure 3). Clearly, six new faces are introduced (four interior and two copies of the original two faces) along with six edges (5 interior & 1 copy) and two new nodes, corresponding to the new d-cells in the arrangement enumerated in Figure 2. Although a chord push induces poor connectivity initially, a sequence of this and other operations will result in improved quality.

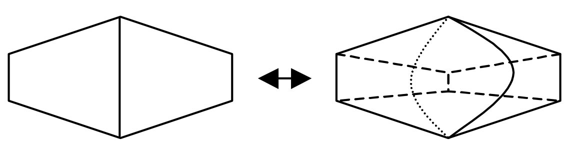

Figure 3: Primal view of chord push operation.

4.2. Hex Push

The hex push operation (see Figure 4) is similar to a chord push, except that, instead of deforming the sheets along a common sheet, they are deformed along a common intersecting sheet pair, or chord. The two vertices, each the intersection between that chord and one of the sheets being deformed, are reversed in their sequence of intersection along the common chord. This appears on both sheets forming the common chord, and on each deformed sheet, as a new line of intersection with the other deformed sheet surrounding the intersection with the common chord.

Again, there are no combinatorial changes to all involved sheets outside the bounding circle on each sheet, therefore this operation is also local.

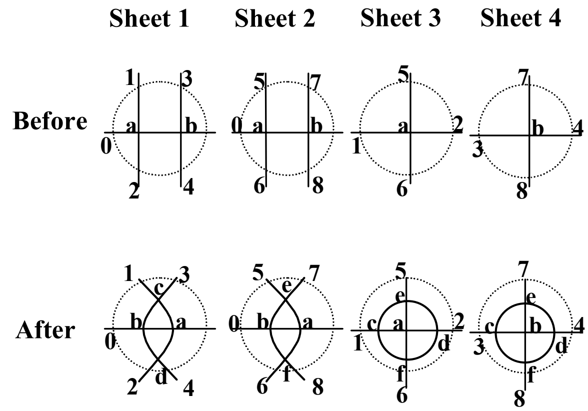

Figure 4: The hex push operation in the dual. Deforming two sheets (sheets 3 and 4) such that they intersect along a common chord (formed by the intersection of sheets 1 and 2). Vertices a and b are reversed in sequence along chord 0, and are surrounded by the new chord of intersection (sheets 3 and 4). This operation produces four new 0-cells (c-f), 12 new 1-cells (cb, ca, bd, ad, eb, ea, bf, af, ec, cf, fd, de), 12 new 2-cells, and four new 3-cells.

We are not certain a hex push is an atomic operation. Repeated applications of chord push operations allows one to come close to surrounding two vertices; we have not yet worked out the final stage of this operation, where the two hexes are pushed through each other. We speculate that this is really the result of somehow collapsing the original hexes and opening new hexes in the appropriate places in the arrangement. This is the subject of further study.

In the primal representation (Figure 5), a hex push swaps the relative locations of the hexes involved in the push. Yet, since the chords of the crossing of dual nodes are anchored on the bounding circle, we create four other crossings as well. These new crossings become four new hexes in the primal mesh. These hexes somewhat “pillow” the newly swapped hexes (Figure 5) all within the bounding faces of this operation. In contrast to the chord push operation, which provably forms elements with negative jacobian quality metrics, the hex push operation forms hexes which do not have intrinsically degenerate quality (though their quality will usually be poor).

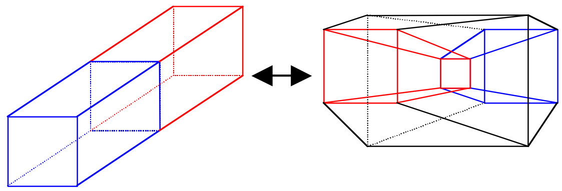

Figure 5: Primal view of hex push operation.

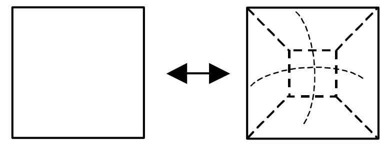

Figure 7: Primal view of (minimal) pillow operation. Single face (left) is pulled apart into two faces (right), producing two hex elements which share five faces.

4.3. (Minimal) Pillowing

In a "pillowing" operation, a new dual sheet is inserted in the dual arrangement such that it induces new vertices. In previous work, the (traditional) pillowing operation has been formulated such that it surrounds an existing vertex in the arrangement [15][17]. However, our view of this operation is a smaller, atomic operation, which is formed along a 1-cell in the arrangement instead of around a vertex. We will sometimes refer to this as a (minimal) pillow operation, to distinguish it from the (traditional) pillowing operation. The dual representation of this operation is shown in Figure 6. The primal corresponding to the (minimal) pillowing operation is shown in Figure 7. Clearly, the quality of the hex elements formed by this operation will be intrinsically degenerate. However, as before, a sequence of operations can be used to improve the quality.

It is straightforward to show that the (traditional) pillowing operation is not atomic; this is shown in Figure 8. Starting with a single dual vertex, a (minimal) pillow is formed along one of the 1-cells to form two new vertices, then a hex push operation is performed between one of the new vertices and the original vertex. The result is the dual arrangement corresponding to a (traditional) pillowing operation performed on the original arrangement. Thus, a (traditional) pillowing operation is a sequence of two atomic operations: a (minimal) pillow, followed by a hex push.

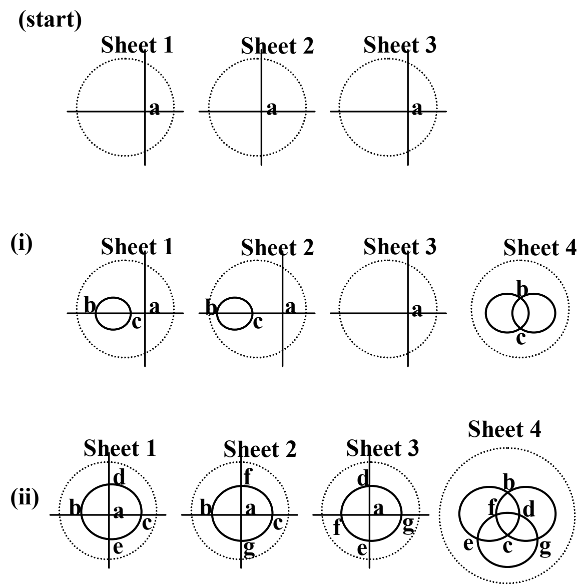

Figure 8: Starting with a single dual vertex (start); add a (minimal) pillow (i); then perform a hex push between vertices a and c (ii). Result is a (traditional) pillow operation as described in [15][17].

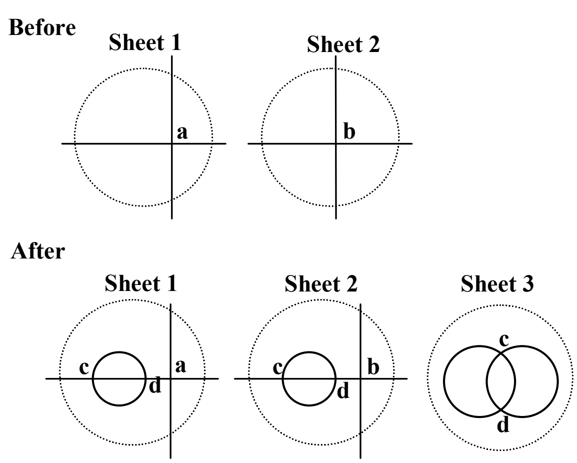

Figure 6: Dual view of a (minimal) pillow operation.

4.4. Face Collapse

We have observed cases where two distinct hex meshes with identical quadrilateral boundaries whose dual arrangements have different numbers of non-pillow hyperplanes, with chords which have different starting and ending quadrilateral faces. For example, the (2,1)-(1,2) flip from [18] starts with five dual sheets in (2,1), but merges two of them to arrive at four dual sheets in (1,2). By definition, transitioning between such meshes using the atomic operations defined thus far is impossible, since none of our operations allows chords to be broken and reconnected, or dual sheets to be broken and reconnected. On the other hand, this type of operation should not be disallowed, as long as it can be done without modifying the quadrilateral boundary, or some bounded region inside the dual arrangement.

We have derived an operation which performs such a modification of the arrangement. This operation has an intermediate state which includes "knife" elements, which are produced when an interior face in the mesh is collapsed by joining two opposite nodes. If the chord corresponding to the face being collapsed is a self-terminating chord (one which does not emerge on the region boundary), the collapse operation produces a single line of self-intersection which terminates on both ends at knife elements. Collapsing the column of elements corresponding to that chord removes the knife elements, changing connectivity only inside the local region. The total operation, consisting of collapsing an entire "ring" of elements, can be considered atomic (because there are no intermediate states with a "valid" arrangement). However, this operation is different from the others, in the sense that it can only be applied to a self-terminating chord (in order to avoid modifying the surface mesh). A ring collapse also results in a merge of two surfaces in the arrangement.

The ring collapse, combined with the two hex push operations, combine to form the results in the transition to the (2,1)-(1,2) flip from [18]. In this case, it happens that one of the "blind" chords produced from the hex pushes is the one collapsed.

4.5. Elemental Operations

We define elemental operations as operations which are not atomic, but which occur often enough that they are useful as distinct sets of atomic operations. We identify three elemental operations: the triple-chord push and the triple-hex push, along with the "traditional" pillowing already described in Section 4.3.

4.5.1. Triple-Chord Push

A triple-chord push modifies the topology of three sheets. In essence, this operation "pulls apart" three faces sharing a common node, such that the result is two hexes sharing three faces. There are many interesting things about this particular arrangement, whose dual and primal representations are shown in Figure 9 and Figure 10, respectively. First, the outer boundary of the polytope shown in Figure 10 has the same number of faces, edges and vertices as a hexahedron, but its dual is not isomorphic to the simple arrangement resulting from the hexahedral elements we have considered up to this point. If it were not for that fact, this arrangement would result in a "parity-flip" operation described in [17]. The composition of a triple-chord push of atomic operations has not yet been derived. We speculate, though, that this operation is composed of three hex pushes (to separate the three faces, and the edges shared by them pairwise, into two sets of three faces each, all sharing the interior node), followed by a "node separate". This last operation might be one more fundamental than a hex push, and could replace the hex push as an atomic operation (meaning that the hex push would be the combination of several chord pushes then a node separate). We plan to investigate this set of operations further in the future.

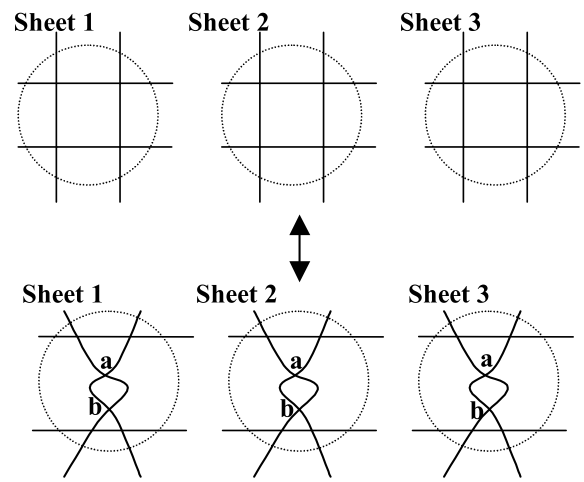

Figure 9: A triple-chord push operation in the dual.

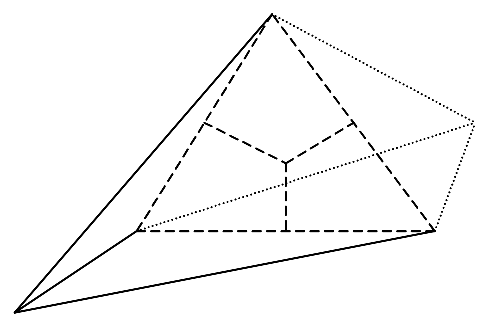

Figure 10: Depiction of the triple-chord push operation in the primal, final state (beginning state is three quadrilaterals sharing a node and sharing edges pairwise).

4.5.2. Triple-Hex Push

Executing on four sheets, a triple-hex push is accomplished by pushing a dual surface through a dual vertex (see Figure 11).

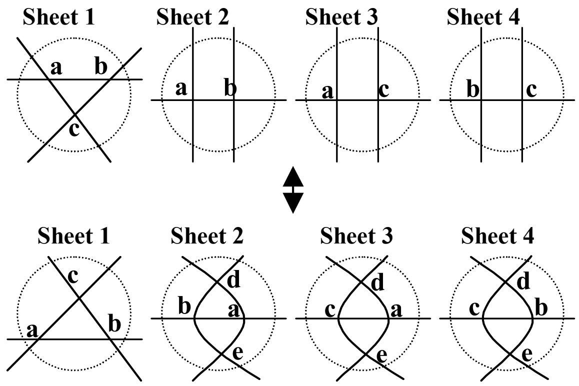

Figure 11: A triple-hex push operation in the dual.

We believe that the triple-hex push can be done as sequence of atomic operations, most likely involving hex push and some form of ring collapse operations. The arrangement resulting from this operation (in the forward direction) is quite similar to that of a sequence of hex pushes, except that such a sequence would create more hexes than **d** and **e** in Figure 11. Therefore, something like a ring collapse is needed for the removal of the extra elements generated.

5. APPLICATION TO FLIPPING OPERATIONS

For brevity, we only state here that all the flipping operations described in [18] can be reproduced by a sequence of the operations described in Sections 4.1-4.4.

6. CONCLUSIONS AND FUTURE WORK

We have described some combinatorial results for dual arrangements resulting from hexahedral meshes. Specifically, we prove that the hex dual is a complex of simple polytopes, and that this complex is also a simple arrangement of hyperplanes (pseudo-hyperplanes, really, since we do not consider geometric embedding of the surfaces and we allow pairs of hyperplanes to intersect more than once). We anticipate these results being useful in proving the completeness of a set of dual-based topology modification operations also described in this paper.
There are several important areas which will receive continued attention. First, the combinatorial results in Section 3 need to be verified in situations where hyperplanes intersect more than once pairwise (most of the classic results were derived under the assumption of single intersections). Furthermore, we intend to use these combinatorial results to characterize allowable modifications to the dual arrangement, and to enumerate the allowable modifications within some bounded region of the arrangement. We anticipate that the allowable modifications will be composed of the operations described in this paper. We also plan to investigate the details of the triple-chord push, due to its interesting combinatorial characteristics.

In addition to future theoretical work, we also plan to develop applications of these operations, specifically for the purposes of hex mesh improvement. Using the non-local information provided by dual sheets, we plan to develop methods for "steering" modifications towards areas of poor quality, enriching or coarsening the topology in order to improve mesh quality locally.

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