BACK TO EDGE FLIPS IN 3 DIMENSIONS

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ABSTRACT

We return to the general edge flip in three dimensions. We demonstrate it is nothing other than a combination of elementary flips. Various properties of this operator are discussed, including conditions that make a flip possible (thus making a set of tets sharing an edge reducible). We discuss also about the existence of other edge based operators and a number of applications are envisaged including an exotic use of some degree of anisotropy.

Keywords: Tetrahedral meshes, edge flip, edge removal (reduction), anisotropy, optimization, delaunisation, boundary integrity.

1. INTRODUCTION AND MOTIVATIONS

The "2-3" flip considers 2 tetrahedra sharing a face and replaces these elements by 3 tetrahedra sharing the edge whose endpoints are opposite the common face, this being made if the resulting pattern is still valid. This flip is the immediate extension of the well-known edge flip (the "2-2" flip or diagonal swaping) in two dimensions. The "2-3" flip was discussed a long time ago and used for various purposes including mesh optimization, boundary enforcement in Delaunay based mesh generation method and some other mesh modifications. The "3-2" flip replacing, when valid, the 3 tets sharing an edge by means of 2 tets sharing a face (that opposite the two endpoints of the above edge endpoints) can be seen as the inverse of the "2-3" flip. The general (of arbitrary order) flip dealing with tets sharing a given edge is the natural extension to three dimensions of the "2-2" flip. The complexity of such flips is cubic (w.r.t. the number of interested tets) while a subtle implementation leads to a almost linear time.

A number of authors, [10], [11], [12], [2], [3], [14], etc., discussed about flips regarding, in specific, the Delaunay triangulation construction in three dimensions. They show that the "2-3" and "3-2" flips can be used to optimize to some degree an arbitrary triangulation with respect to the Delaunay criterion.

In this paper we discuss about the general flip and we show that it is a combination of elementary "2-3" flips together with a "3-2" or a "4-4" flip. This general flip can be used to remove an edge in a mesh. We give some conditions that make this removal (reduction) possible. Also we discuss if there is any other type of flips and, to conclude, we indicate various applications of such flips and we propose an anisotropic point of view.

2. BACK TO THE "2-3" FLIP

The "2-3" flip considers the polyhedron made up of 2 tets sharing a face. If this polyhedron is convex, Figure 1 (left), or not, Figure 1 (right), there exists an alternate tet configuration made up of 3 tets which covers the same volume or such a solution is not valid. The

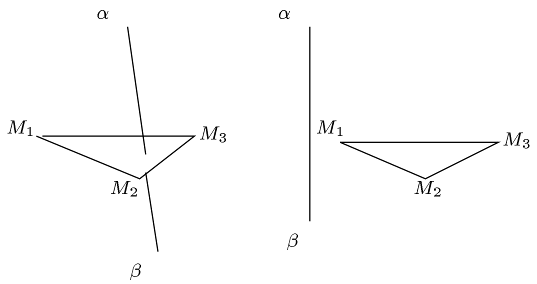

Figure 1. *Schematic of the configuration related to the 2 tets $M_1M_2M_3\alpha$ and $\beta M_1M_2M_3$. Left, edge $\alpha\beta$ cuts the triangle corresponding to the common face, right, the case is not convex.*

initial situation reads $K_1 = [\, M_1M_2M_3\alpha \,]$ and $K_2 = [\, \beta M_1M_2M_3 \,]$. The resulting situation, when valid,

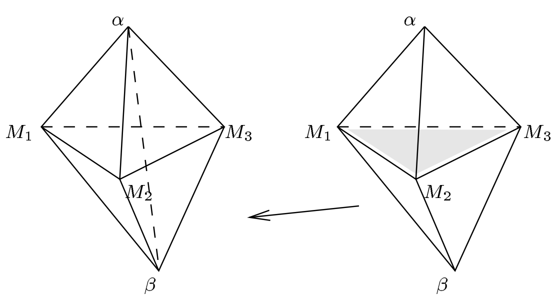

Figure 2. *The polyhedron made up of the 2 tets $M_1M_2M_3\alpha$ and $\beta M_1M_2M_3$, right, is convex and can be replaced by means of the 3 tets $M_1\beta\alpha M_2$, $M_2\beta\alpha M_3$ and $M_3\beta\alpha M_1$, left.*

reads : $K_1^* = [\, M_1\beta\alpha M_2 \,]$, $K_2^* = [\, M_2\beta\alpha M_3 \,]$ and $K_3^* = [\, M_3\beta\alpha M_1 \,]$. In other words, we obtain a *shell* made up of 3 elements, such a set being defined now.

Definition 1. Given an edge, a *shell* is the polyhedron made up of the tets sharing this edge. The common edge is the *generating edge* of the shell. The vertices other than the edge endpoints constitute the *generating polygon* of the shell. □

Note that the above polygon is ordered and, in general, non planar. Note also that only *closed* shells are discussed in the paper, eg the edge is fully surrounded by tets.

From the topological point of view, the "2-3" flip removes one edge and creates one face. To validate such a flip, one has to check the positiveness of the 3 resulting tets (in other words, the polyhedron is convex or $\alpha\beta$ passes through triangle $M_1M_2M_3$).

3. BACK TO THE "3-2" FLIP

The "3-2" flip considers the polyhedron made up of 3 tets sharing an edge. If the so-defined polyhedron is convex, Figure 3 (left), or not, Figure 3 (right), there exists a alternate valid configuration made up of 2 tets which covers the same volume, Figure 4, or this case is not valid. Note that in a non-convex case, edge $\alpha\beta$ does not cut the triangle whose vertices are other than α and β while the supporting line of $\alpha\beta$ cuts it. In other words, the plane of this triangle separates α from β or not. The configuration where the plane is not a separation plane is called a *perfect Christmas tree* and cannot be remeshed.

Definition 2. A shell is a *perfect Christmas tree* if its generating polygon is planar and does not separate the two endpoints of its generating edge. □

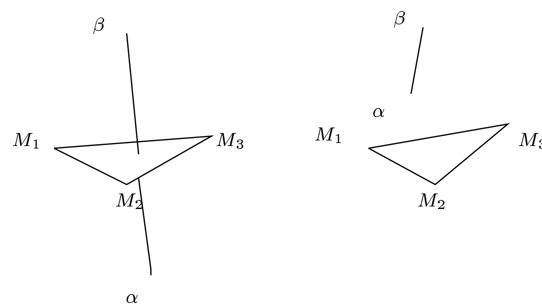

Figure 3. *Schematic of the configuration related to the polyhedron made up of the 3 tets $M_1\alpha\beta M_2$, $M_2\alpha\beta M_3$ and $M_3\alpha\beta M_1$. left, edge $\alpha\beta$ cuts the triangle based on the 3 vertices other than α and β, right, the configuration is not convex.*

The initial situation reads : $K_1 = [\, M_1\alpha\beta M_2 \,]$, $K_2 = [\, M_2\alpha\beta M_3 \,]$ and $K_3 = [\, M_3\alpha\beta M_1 \,]$. The resulting situation, when valid, reads : $K_1^* = [\, M_1M_2M_3\beta \,]$, $K_2^* = [\, \alpha M_1M_2M_3 \,]$. From the topological point of view, the "3-2" flip removes one face and creates one edge. To validate such a flip, one has to check the positiveness of the volume of the two resulting tets (in other words, $\alpha\beta$ passes through triangle $M_1M_2M_3$).

At a latter stage, we will discuss conditions that make this flip possible.

4. THE NECESSARY "4-4" FLIP

A 4-tet shell simply reads (after permuting the indices of its generating polygon) : $K_1 = [\, M_1\alpha\beta M_2 \,]$, $K_2 = [\, M_2\alpha\beta M_3 \,]$, $K_3 = [\, M_3\alpha\beta M_4 \,]$ and $K_4 =$

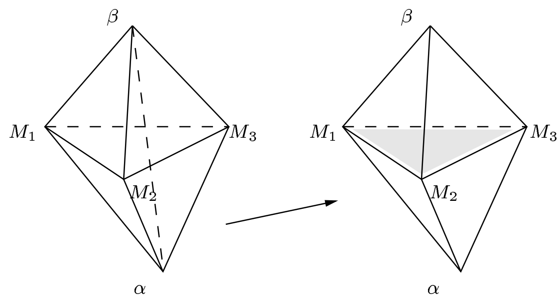

Figure 4. *The polyhedron made up of the 3 tets $M_1\alpha\beta M_2$, $M_2\alpha\beta M_3$ and $M_3\alpha\beta M_1$, left, is convex and can be remeshed by means of the 2 tets $M_1M_2M_34\beta$ and $\alpha M_1M_2M_3$, right.*

$[\,M_4\alpha\beta M_1\,]$. Note that such a shell can be a perfect Christmas tree. Nevertheless, the generating polygon is not necessarily planar and, even planar, is not necessarily convex while the shell exists and is valid.

Definition 3. A shell is a *Christmas tree* if no mesh of its generating polygon exists which separates α from β. □

A shell is not a Christmas tree if a mesh of its generating polygon for which all the triangles are visible by α and β exists and, as a consequence, edge $\alpha\beta$ cuts this (separation) mesh at a unique point (inside a triangle or on one triangle edge).

A perfect Christmas tree is then a peculiar Christmas tree. A non-Christmas tree shell with a planar generating polygon has a separation plane for α and the other points and another separation plane (possibly the same) for β and the other points, thus the definition is consistent.

4.1 Reducing a 4-tet shell using “2-3” and “3-2” flips

As K_1 and K_2 share face $(\alpha\beta M_2)$, a “2-3” flip can be envisaged. If possible, a brute force reading leads to $T_1^* = [\,\alpha M_1M_3\beta\,]$, $T_2^* = [\,\beta M_1M_3M_2\,]$ and $T_3^* = [\,M_2M_1M_3\alpha\,]$, to which we must add K_3 and K_4. Edge $\alpha\beta$ is now common to 3 elements, eg T_1^*, K_3 and K_4. A “3-2” flip can be then envisaged. From $T_1^* = [\,M_1\alpha\beta M_3\,]$, $K_3 = [\,M_3\alpha\beta M_4\,]$ and $K_4 = [\,M_4\alpha\beta M_1\,]$,, we find, when valid, $T_4^* = [\,M_1M_3M_4\beta\,]$ and $T_5^* = [\,\alpha M_1M_3M_4\,]$. The resulting tets are then $T_2^* = [\,M_1M_2M_3\beta\,]$, $T_3^* = [\,\alpha M_1M_2M_3\,]$, $T_4^* = [\,\alpha M_1M_3M_4\,]$ and $T_5^* = [\,M_1M_3M_4\beta\,]$. In other words, the M_i’s polygon is now meshed by means of 2 triangles, eg $M_1M_3M_4$ and $M_1M_2M_3$. The solution is made up of 4 tets formed by joining α and β with these 2 triangles.

While edge $\alpha\beta$ is no longer a mesh edge, we say we have reduced the given shell (thus the term shell reduction).

Using a “2-3” flip on elements K_2 and K_3, we obtain an alternate solution based on the two alternate faces covering the polygon, eg $M_1M_2M_4$ and $M_2M_3M_4$. Moreover, there is no more solutions, another “2-3” flip leading to the same combinations.

4.2 Impossible reduction of a 4-tet shell using these flips

There is a configuration for which exists an alternate mesh which is not obtained in the above way (eg by means of “2-3” and “3-2” flips).

Let us consider a convex shell (thus an alternate mesh clearly exists) and let us assume that edge $\alpha\beta$ cuts segment M_1M_3 together with segment M_2M_4, then no “2-3” flip is valid. As a consequence, it is strictly needed to define a “4-4” flip which directly constructs the solution whose existence is known in advance.

4.3 The “4-4” flip

Here we follow a simple idea (used for the higher order flips). We consider the M_i’s polygon, we mesh it by means of triangles and, finally, we join these triangles with vertices α and β. Therefore, there are at most 2 solutions.

5. GENERAL HIGHER ORDER FLIPS

Such flips consider shells made up of n elements. Such a shell reads $K_i = [\,M_i\alpha\beta M_{i+1}\,]$ with $i = 1, ..., n$ and $M_{n+1} \equiv M_1$.

5.1 Combination of elementary flips and direct flip

Let C_n be a n tet shell, with evident while abusive notations we can write $C_n = “2-3'' \cup C_{n-1}$, as soon as $n > 4$, this being possible as one “2-3” flip is valid. Therefore, if $n-4$ such flips are valid, one can write $C_n = (n-4)“2-3'' \cup C_4$, and an ultimate “4-4” flip, if valid, removes edge $\alpha\beta$ from this last shell. Using the same argument as that for a “4-4” flip, we can say that $C_n = (n-3)“2-3'' \cup C_3$, is not, in general, a way to access the expected solution, in specific, while knowing it exists.

However, the computer writing, for $n \geq 4$, is much more faster (while more technical) if we are given in advance all the candidate solutions. This reduces to enumerate all the *a priori* possible remeshing of a polygon with n sides, see Figure 5 for $n = 5$ and Figure 6 for $n = 6$. Each triangle in these remeshing is then connected with α and β to constructing the desired tets. The direct flip is based on the data of the catalogue of all the *a priori* possible solutions and leads to pick up in this series one or more instead of applying a number of “2-3” flips so as to obtain a 3 (or 4)-tet shell.

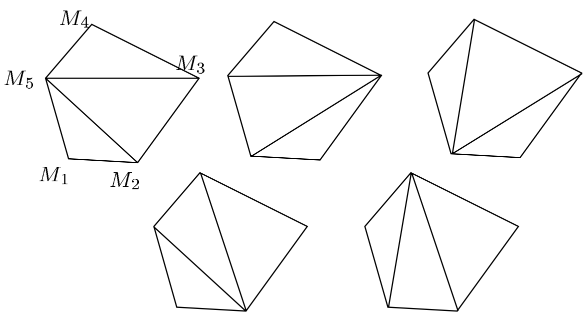

Figure 5. *The 5 solutions related to a 5-side generating polygon.*

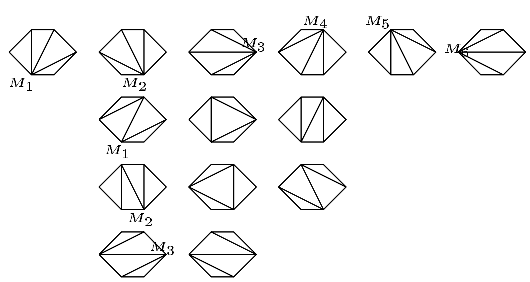

Figure 6. *The 14 solutions for a 6-side generating polygon. Top, 3 edges are incident to each M_i, then, 2 edges are incident to M_1 to M_2 and to M_3 where only 2 cases appear, then, the other cases, related to M_4 and M_5 have already being seen.*

5.2 Is a shell reducible ?

Definition 4. A shell is *reducible* if there exists a remeshing of the corresponding polyhedron where its generating edge is no longer a tet edge (it has been removed). □

On the fly, we already meet some conditions that make a shell reducible or not.

- A perfect Christmas tree is not reducible.
- For $n = 3$, a shell other than a (perfect) Christmas tree is reducible (indeed, edge $\alpha\beta$ cuts the generating triangle and, therefore, as α and β see this triangle, the 2 solution tets are valid).

To continue the discussion, we have to consider the general case ($n > 3$ and a non planar polygon) and to find conditions that make the shell reducible. We will demonstrate that :

- if the polygon is planar, all shells other than a (perfect) Christmas tree are reducible,
- for $n = 4$, the same occurs for a shell other than a Christmas tree provided an additional condition,
- and this extents to an arbitrary n.

Planar polygon. For a planar generating polygon, the proof is obvious. Any triangular meshes of the polygon is such that each of its triangles is visible by α and by β (as the plane separates these two points), thus the resulting tets are valid (positive volumes). Note that the fact that the polygon is convex or not is not an issue. This ends the proof of reductibility for a shell with a planar polygon.

Now, we will demonstrate that this proof can be obtained by means of a number of “2-3” flips ended by an unique “3-2” or “4-4” flip. Let P be the intersection point of segment $\alpha\beta$ with the plane supporting the polygon. If this polygon has more than 5 sides, there exists one index i such that triangles $M_{i-1}M_iM_{i+1}$ and $M_{i+1}M_{i+2}M_{i+3}$ (after a modulo) lie inside the polygon. As the intersection of these 2 triangles is point M_{i+1}, point P cannot belong to these 2 triangles. As a consequence, applying a “2-3” flip reduces by 1 the polygon. For example, if triangle $M_{i-1}M_iM_{i+1}$ does not contain P, a “2-3” flip applied to tet $M_i\beta\alpha M_{i+1}$ and $M_{i-1}M_i\beta\alpha$ results in tets $M_iM_{i-1}M_{i+1}\beta$ $\beta M_{i-1}M_{i+1}\alpha$ $\alpha M_{i-1}M_{i+1}M_i$ and, therefore, only one of these tets includes edge $\alpha\beta$. Note that this flip removes triangle $M_i\beta\alpha$ and forms edge $M_{i-1}M_{i+1}$. This operation is possible because triangle $M_{i-1}M_iM_{i+1}$ separates α from β and edge $M_{i-1}M_{i+1}$ cuts triangle $M_i\beta\alpha$. Thus, point M_i is no longer a vertex of the generating polygon. We then repeat the same construction until a polygon with 3 or 4 sides remains where an ultimate “3-2” or “4-4” flip applies.

Non planar polygon. In this case, the proof for the reductibility is obvious while it is more subtle to see that a combination of flips gives the solution (if valid).

The reductibility results from the definition of what a Christmas tree is. For shells other than such a tree, the existence of a mesh such that α and β are visible by the triangles covering the polygon guarantees that the corresponding tets are valid. Thus, edge $\alpha\beta$ is no longer a mesh edge and the reductibility holds.

To see that the solutions results from a combination of flips, we first look at the case $n = 4$ before noticing that the general case reduces to the same simple situation.

For $n = 4$, assumed a non-Christmas tree case, there exists a mesh made up of 2 triangles visible by α and β. The generating edge necessarily cuts this mesh. If the intersection falls inside one of these triangles the other triangle allows for a "2-3" flips resulting in a 3-tet shell, which is necessarily convex, thus one "3-2" flips gives a solution. If the intersection is on the edge common to these 2 triangles, a "4-4" flip gives a solution. Therefore, for $n = 4$, a shell is reducible.

For an arbitrary n, if the shell is reducible it exists a triangular mesh of the generating polygon which separates α from β. In this mesh exists (see below) a triangle made up of three consecutive M_i's, say $M_{j-1}M_jM_{j+1}$, that does not cut $\alpha\beta$ (and separates α from β). This property makes the "2-3" flip removing triangle $\beta\alpha M_j$ possible and reduces by 1 the size of the generating polygon (indeed, vertex M_j is no longer a member of the updated polygon). As the reduced shell remains reducible (with the remaining triangles of the initial triangular mesh of the generating polygon), the same applies for the various reduced configurations. Once these flips have been applied, it remains a shell where $n = 3$ (thus reducible) or $n = 4$ for which the above discussion applies. To conclude, a shell where $n > 4$ is reducible as soon as it is not a Christmas tree.

To complete the proof, it is needed to see that above triangle $M_{j-1}M_jM_{j+1}$ exists. Let us consider a plane orthogonal to $\alpha\beta$ cutting this segment. The projection of the polygon onto this plane is a simple polygon (eg. non self-intersecting) surrounding $\alpha\beta$. Indeed, this polygon is star-shaped with respect to the intersection point of the plane with segment $\alpha\beta$ because all the tets in the shell with α and β have a positive volume and the projection onto the plane of these tets maintains the orientation of the boundary of the polygon with respect to segment $\alpha\beta$.

In other words, the correctness of the orientation of the projected polygon holds if the volume of the $M_i\alpha\beta M_{i+1}$'s has, for each of these tets, the same sign than the volume of the tets $\widetilde{M}_i\alpha\beta\widetilde{M}_{i+1}$ where $\widetilde{M}_i$ is the projection of M_i onto the plane. At a factor 6, we have $V = \vec{\beta\alpha} \,.\, (\, \vec{\beta M_i} \wedge \vec{\beta M_{i+1}} \,)$, then we compute $\widetilde{V} = \vec{\beta\alpha} \,.\, (\, \vec{\beta\widetilde{M}_i} \wedge \vec{\beta\widetilde{M}_{i+1}} \,)$. As $(\, \vec{\beta\widetilde{M}_i} \wedge \vec{\beta\widetilde{M}_{i+1}} \,) = (\, \vec{\beta M_i} + \vec{M_i\widetilde{M}_i} \,) \wedge (\, \vec{\beta M_{i+1}} + \vec{M_{i+1}\widetilde{M}_{i+1}} \,)$, we have, for $\widetilde{V}$, 4 contributions, eg $\vec{\beta\alpha} \,.\, (\, \vec{M_i\widetilde{M}_i} \wedge \vec{\beta M_{i+1}} \,)$ which is null as $\beta\alpha$ is parallel to $\vec{M_i\widetilde{M}_i}$, $\vec{\beta\alpha} \,.\, (\, \vec{\beta M_i} \wedge \vec{M_{i+1}\widetilde{M}_{i+1}} \,)$ which is null as $\beta\alpha$ is parallel to $\vec{M_{i+1}\widetilde{M}_{i+1}}$, and $(\, \vec{M_i\widetilde{M}_i} \wedge \vec{M_{i+1}\widetilde{M}_{i+1}} \,)$ which is also null as the 2 vectors involved are parallel. Therefore $\widetilde{V} = V$, which ends the proof about the orientation of the initial polygon and the projected polygon.

Thus, it is sufficient to analyse the (planar) projected configuration. As $n > 4$, there are at least 2 triangles based on three consecutive vertices and edge $\alpha\beta$ cannot cut both of them. Therefore, one of these triangles allows for the solution.

In other words, if a shell is reducible, its reduction can be obtained using a number of "2-3" flips with a "3-2" or a "4-4" flip.

Note that in this reasoning we have considered the solution (the above triangular mesh) to determine the necessary "2-3" flips. Thus, these flips are not known in advance and the complexity of the method relies in effectively finding what flips must be applied.

A couple of remarks. All the previous discussion (apart for the non reductibility) is no longer valid, in practice, if one likes to include quality concerns (and not only a volume check). Also, a more restrictive definition of a Christmas tree can be advocated, eg, a Christmas tree occurs when there are not two planes (and not only a non planar triangular mesh) that separates, one α and the other points, the other β and the other points.

6. COMPLEXITY ISSUES

To discuss the complexity of the flips, we first recall the number of possible triangulations and the number of different triangles covering the generating polygon of an arbitrary shell. Then, we turn to the theoretical complexity of a flip (for shell reduction or for mesh optimization) before restricting ourselves to the actual cases where n is relatively small (up to 6 or 7).

6.1 Number of solutions versus n

Table 1 gives N_n, the number of possible triangulations as a function of n. It also gives Tr_n the number of different triangles in one possible triangulation. This concerns the topological point of view and not any validity aspect.

n	3	4	5	6	7	8	9	10
N_n	1	2	5	14	42	132	429	1 430
Tr_n	1	4	10	20	35	56	84	120

Table 1. Number of possible triangulations versus the number of sides of the generating polygon.

We have $N_n = Cat(n-1)$ where the Catalan number is involved which reads $Cat(n) = \frac{(2n-2)!}{n!(n-1)!}$. On the other hand, $Tr_n = C_n^3$ holds.

While being a classical result, we have pleasure to establish the value of the Catalan number. To this end, let us consider a (ordered) series of objects simply denoted as $(1\,2\,3 \ldots n)$. Let S_n be the number of combinations of the various different grouping of those objects. To find a recursion about S_n, we can write as a first case the grouping of (1) with the $n-1$ other objects $(2\,3 \ldots n)$: $G_1 = (1)(2\,3 \ldots n)$, as a second case, we consider the grouping of $(1\,2)$ with$(3\,4 \ldots n)$, eg $G_2 = (1\,2)(3 \ldots n)$, and, ..., as case # i, we have $G_i = (1\,2 \ldots i)(i+1\; i+2 \ldots n)$. Thus, $S_n = G_1 + G_2 + \ldots + G_{n-1}$, in other words, $S_n = S_1 S_{n-1} + S_2 S_{n-2} + S_3 S_{n-3} + \ldots + S_{n-1} S_1$, and then $S_n = \sum_{i=1}^{n-1} S_i\, S_{n-i}$ holds. To exhibit an explicit writing for S_n, we consider the polynomial associated with S_n, eg $F(z) = \sum_{i=1}^{\infty} S_i\, z^i$. A simple calculation shows that $F(z)\,F(z) = F(z) - z$, from which we have $F(z) = \frac{1-\sqrt{1-4z}}{2}$.

Let us expand $\sqrt{1-4z}$ nearby 0. To this end, we look at the expansion of $(1+\varepsilon)^m$ for a small ε. We have $(1+\varepsilon)^m = 1 + m\varepsilon + \frac{m(m-1)}{2}\varepsilon^2 + \frac{m(m-1)(m-2)}{3!}\varepsilon^3 + \ldots + \frac{m(m-1)(m-2)\ldots(m-i+1)}{i!}\varepsilon^i + \ldots$.

For $m = \frac{1}{2}$ and $\varepsilon = -4z$, the coefficient of the term in z^i, for i not 0, reads : $\frac{1}{i!}\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\ldots(\frac{1}{2}-i+1)(-4)^i$, after factorizing $\frac{1}{2}$, we have : $\frac{1}{i!}(\frac{1}{2})^i(-4)^i\,(1)(1-2)(1-4)\ldots(1-2i+2)$, or again $-\frac{1}{i!}2^i\,(1)(3)(5)\ldots(2i-3)$, or, finally, $-\frac{1}{i!}2^i\,\frac{(2i-2)!}{(2)(4)(6)\ldots(2i-2)}$, or $-2\,\frac{1}{i!}\,\frac{(2i-2)!}{(i-1)!}$. As in $F(z)$ the coefficient of the term in z^0 is 1, we have $F(z) = \sum_{i=1}^{\infty} \frac{(2i-2)!}{(i-1)!\,i!}\,z^i$, therefore, after identification, $S_n = \frac{(2n-2)!}{(n-1)!\,n!}$ holds which gives the value of $Cat(n)$ (eg S_n).

6.2 Number of different triangles versus n

We have $Tr_n = C_n^3 = \frac{n\,(n-1)\,(n-2)}{6}$, and, *a priori*, the method is cubic in n thus in the number of needed validity checks.

6.3 Theoretical complexity

The complexity involves three parts, one related to constructing the list of the N_n candidate solutions, the second related to exhibit the number of different triangles for the generating polygon, Tr_n and, finally, the cost needed to validate such or such solution with respect to the purpose (reductibility, optimization or whatever).

N_n, the number of candidate triangulations of a shell increases as an exponential in n, thus the cost to exhibit these triangulations is *a priori* non polynomial. Therefore, optimizing a shell is non polynomial. However, if we consider only the triangulations star-shaped with respect to one of the vertices in the polygon, N_n becomes linear in n.

Edge flips for reduction purpose has a non-polynomial cost. Indeed, the triangular mesh solution gives the order in which the "2-3" flips must be applied and this solution must be exhibited among the N_n cases. However, in the planar case, the cost is only quadratic, in fact, we can consider every three consecutive vertices in the polygon leading to a triangle which allows for a "2-3" flip and the resulting polygon is reduced by one, then, the same applies.

In the non-planar case, this simple procedure does not apply because it is not proved that reducing by one a reducible shell results in a shell which is still reducible. In fact, simple cases can be constructed where applying a flip may results in a non-flippable reduced shell (while being reducible before). This suggests defining an order when choosing a flip to maintain a reducible shell at each step until the final reduction.

6.4 Computer issues

Reducing the effective cost of a flip is achieved by a rapid rejection of as many *a priori* candidate solutions as possible when evaluating the various cases. A simple idea allows for this. We just have to classify the candidate triangles as a function of their frequency, Figure 7. Therefore, a negative analyse of one (at most 2) tets related to one such triangle allows to immediately reject a number of cases. In the example in the figure, rejecting triangle t_1 in case $i)$ leads to reject case $iv)$ and thus triangle t_9 (and related tets) are never considered. Moreover, instead of considering case $i)$ and then the next case, triangles will be checked following the above classification. Clearly, the higher order the shell the higher benefit. In this way, analyzing all the possible cases is unlikely to be possible. Actually, only shell of order up to 6 are of real interest and the cost is neglictible. In this case, the

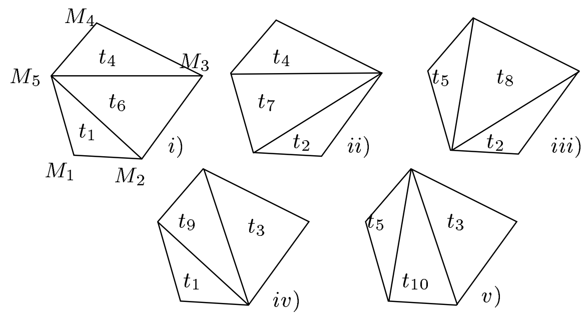

Figure 7. *The 5 solutions related to a 5-side polygon with frequency classification.*

full list of the candidate solutions is made in advance (by hand) and thus is a null cost process (while being non polynomial in general).

7. OTHER FLIPS ?

As soon as the generating polygon is not a planar polygon and has at least 4 points, it is possible to construct tets with positive volume whose vertices are 4 of these points. The simpler example concerns a 4-tet shell.

If tet $[M_1 M_2 M_3 M_4]$ is positive, this shell can be possibly written as a polyhedron with 5 tets, $[M_1 M_2 M_3 M_4]$ $[M_1 M_2 M_4 \beta]$ $[M_2 M_3 M_4 \beta]$ and $[\alpha M_1 M_2 M_3]$ $[\alpha M_1 M_3 M_4]$.

If $[M_1 M_2 M_3 M_4]$ is negative, the writing is $[M_4 M_2 M_3 M_1]$ $[M_1 M_2 M_3 \beta]$ $[M_1 M_3 M_4 \beta]$ and $[\alpha M_1 M_2 M_4]$ $[\alpha M_2 M_3 M_4]$.

Is it a new flip ? No, it is not, to be convinced, consider the last remeshing and compare it to the second writing already seen (in Section about the "4-4" flip) : $[\alpha M_2 M_3 M_4]$ $[M_1 M_2 M_4 \beta]$ $[M_2 M_3 M_4 \beta]$ and $[\alpha M_1 M_2 M_4]$. A "2-3" flip applied to $[M_1 M_2 M_4 \beta]$ and $[M_1 M_3 M_4 \beta]$ with the common face $M_1 \beta M_4$ results in the 3 tets $[M_4 M_2 M_3 M_1]$ $[M_1 M_2 M_3 \beta]$ $[M_1 M_3 M_4 \beta]$.

Thus the 5-tet solution is obtained after applying a "2-3" flip (which is often possible as the 4 M_i's are not planar) to 2 tets in the classical solution. So it is for $n > 4$.

8. FACE FLIPS ?

Edge flips remove an edge, a question is then to decide if exists similar transformations which remove a face (which could be seen as a *face removal* operator).

An idea is to find such a transformation as the inverse of an edge flip. Indeed, the "2-3" flip , inverse to a "3-2" flip, seems to be an example of such a transformation and thus each edge flip, $n > 3$, should have a corresponding face flip.

Actually, designing a face flip reduces to find 2 points α and β and a polygon made up of faces that see these 2 points. This implies some properties about these faces.

Let α be a vertex of tet K_1 and let f_α be its opposite face in this tet ($K_1 = [\alpha f_\alpha]$). Let K_2 be the tet sharing face f_α with K_1. Point β, opposite this face in K_2 defines with α segment $\alpha\beta$. If $\alpha\beta$ cuts f_α, we return to a known case (2 tets sharing a face) where a "2-3" flip applies which removes face f_α and construct edge $\alpha\beta$. If segment $\alpha\beta$ does not cut f_α, we find the tets cut by this segment. After some conditions we return to a pattern that can be seen as the inverse of an edge flip.

Above conditions reduces to one condition :

- either there is only one tet face cut by $\alpha\beta$,
- or $\alpha\beta$ cuts one edge, ab, common to a number of tets whose other vertices are α and β.

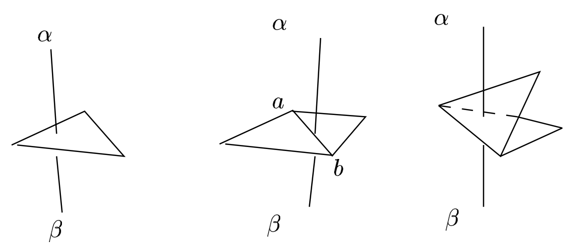

Figure 8. *Some local situations for segment $\alpha\beta$. Left, the 2 possible cases, i) 2 elements exist sharing a face cut by $\alpha\beta$ such that the opposite vertex is α in one and β in the other, ii) segment $\alpha\beta$ cuts an edge which defines a shell where the other vertices are α and β. Right, segment $\alpha\beta$ cuts several tets thus making this case not candidate for a "face" flip.*

As a consequence, the related polygon is either made up of the vertices of the cut face or made up of the vertices of the 2 faces sharing edge ab.

In other words, all the other cases are not candidate to a flip thus reducing the field of applications of such an (face flip) operator. For completeness, however, it must be noticed that some peculiar meshes exhibit candidate cases.

To end, let us remark in the case of a "2-3" flip, in a convex case, that the polygon is *a priori* made up of the ordered list of the vertices of the common face but, after some conditions, it could be augmented by some neighbouring faces so as to arrive to a larger pattern, Figure 9. The aim is here to increase a quality criterion, let us think to a case where $\alpha\beta$ cuts triangle $M_1M_2M_3$ close to edge M_2M_3.

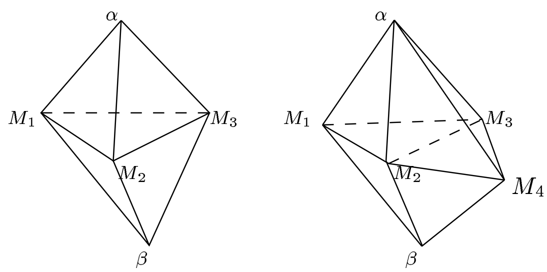

Figure 9. *Enlarging the polygon candidate for a "2-3" flip. While a priori reduced to 3 vertices, another vertex is added together with 2 tets having the requested property.*

9. ANISOTROPIC FLIPS

In this section, we introduce an exotic use of edge flips in an anisotropic context. To this end and for simplicity, we return to the "2-3" flip. We extent this operator to an anisotropic context. Then we show how this simple operator applies in the crucial boundary enforcement step in a Delaunay based mesh generator (while being usable for other purposes).

From the topological point of view, nothing new in this case. The simple underlying idea is to govern the flip by introducing an anisotropic quality function.

Therefore we are concerned with an "optimization" problem where the quality function must be defined. Let us recall a quality function used for mesh optimization in a classical (eg isotropic) case. The function which has our favor, for such a point of view, reads $Q_K = coef\frac{h^3}{V_K}$, where $h = \sqrt{\sum_{i=1}^{6} L_i^2}$ with L_i the length of edge # i in tet K, V_K the volume of tet K and $coef$ a normalization coefficient.

Let us develop an anisotropic quality function based on the above expression. To this end, we introduce $\mathcal{M}$ the matrix corresponding to the anisotropic metric in hand, then $Q_K^{\mathcal{M}} = coef\frac{h^3}{V_K^{\mathcal{M}}}$, with $V_K^{\mathcal{M}} = Det(\mathcal{M})\,V_K$ and $h = \sqrt{\sum_{i=1}^{6}(L_i^{\mathcal{M}})^2}$ where $L_i^{\mathcal{M}} = \sqrt{< A_i, \mathcal{M}A_i >}$, symbol $< .,. >$ standing for the dot product while A_i is the vector related to edge # i in tet K. For the sake of simplicity, we assume the matrix to be constant over K.

The envisioned application concerns the tedious boundary enforcement in a Delaunay based mesh generation method. It is known that such a process is mainly based on edge flips (at least in the approach we have proposed a number of years ago). The idea (which is so simple but took about 10 years to be matured !) trivially consists in governing such flips using a local and temporary defined metric based on the edge we like to create.

The above anisotropic metric (matrix) is then defined in accordance. Let us recall that $\mathcal{M}$, a general metric, reads as a 3×3 symmetric positive definite matrix which can be also written as $\mathcal{M} = \mathcal{R}\,\Lambda\,{}^t\mathcal{R}$, with $\mathcal{R}$ an orthogonal matrix and Λ a diagonal matrix with positive entries.

Therefore, if AB is the sought edge, we define the metric as

$$h = \|AB\|$$

$$\Lambda = \begin{pmatrix} \frac{1}{h^2} & 0 & 0 \\ 0 & \frac{1}{(k\,h)^2} & 0 \\ 0 & 0 & \frac{1}{(k\,h)^2} \end{pmatrix}, \qquad (1)$$

$d_1 = \frac{AB}{h}$ and d_2 such that $< d_1\,, d_2 >= 0$ and $\|d_2\| = 1$ and, finally, $d_3 = \frac{d_1 \wedge d_2}{\|d_1 \wedge d_2\|}$ with $k << 1$. From $d_1\,, d_2$ et d_3, we define

$$\mathcal{R} = {}^t\,(d_1, d_2, d_3)\,.$$

Thus $\mathcal{M}$ is well defined and enjoys the good properties.

Note that this definition allows to artificially make points A and B closer while the other points are, temporarily, made farther. Also the direction of the vector supporting edge AB is favored.

In [9] (in french) we show how this trivial method is used and reduces the cost of the boundary enforcement step included in a Delaunay based mesh generation method.

10. APPLICATIONS

Edge flips allow for a variety of applications among which we select what follow.

10.1 Tet mesh optimization

Edge flips (together with node repositioning) is one of the tool used for tet mesh optimization purpose, [1]. All (internal) edge are considered as candidate for a flip.

Criterion is no longer the volume positiveness but a quality function. As already mentioned, a rapid rejection of the unlikely suitable solutions is crucial leading to a rather effective method with a low cost (in specific as compared with the cost of node repositioning). In this way, optimizing a large number of tets takes only a couple of seconds (0.75 sec for a mesh with $85\,764$ tets, 4.00 sec. for $430\,033$ tets in our computer implementation).

10.2 Tet mesh delaunisation

In this section, we turn to two *a priori* different questions. One could be "is it possible to replace the Delaunay kernel, [6], by just locally splitting the element within which falls the point under insertion and then applying a series of edge flips". The other could be to see "if edge flips allow to make an arbitrary mesh a Delaunay mesh".

Direct point insertion *plus* **edge flips.** Such a method perfectly runs in two dimensions. Inserting a point reduces to find the triangle(s) within which the point falls, split this triangle(s) into 3 (4) sub-triangles and apply a series of edge flips until the Delaunay criterion is locally satisfied. Is it the case in three dimensions ? This problem has been discussed in [12] and [11] which consider the configurations of 5 distinct non-coplanar points. In [11], an algorithm using low order edge flips is proposed which assumes that the point insertion follows a peculiar order: the current point to be inserted must be outside the convex hull of the already inserted points. In the general case, this problem appears to be still open. However we think that this problem can be translated in another one which says that some point to point connections are missing while some others are to be deleted. The idea could be to remove the extra connections while recreating the missing ones using flips. The key would be to prove that flips never lead to a non Delaunay configuration which is no longer "flippable".

Delaunisation of an arbitrary mesh. We are given an arbitrary mesh (eg non Delaunay) and we like to apply a number of edge flips so as to arrive to a Delaunay mesh[1].

[1]Thus the neologism "Delaunisation".

This is known in two dimensions for a *triangulation*[2] and it is also true for a *mesh* not for the Delaunay criterion but for a constrained variant of this property. Is it the case in three dimensions ?

In [10] is given a 3D example for which using "2-3", "3-2" and "4-4" flips to fullfil the Delaunay criterion does not complete a Delaunay triangulation. The best we can do is to conjecture that applying such flips even with a Delaunay criterion violation result in a Delaunay triangulation.

The same question for a meshing problem is much more tedious since constrained entities (must) exist. In this case, it is not safe to formulate any conjecture.

In other words, this question seems to be still open.

10.3 Boundary enforcement in a Delaunay based mesh

As partially evocated, a natural use of edge flips is to remove edges (and faces) in a tet mesh and to create alternate edges and faces. This is the key point (while being not sufficient) in the method we proposed in the mesh generator developed at INRIA, [5], [7].

10.4 Anisotropic meshing

In this application, we are given a classical (thus isotropic) tet mesh and we like to introduce some degree of anisotropy in some regions, [13]. The "2-3" flip appears to be attractive to handle such a problem. Let us consider 2 adjacent tets where the common face has a nice quality (following an isotropic quality function). Clearly, in this pattern (common face $M_1M_2M_3$ and opposite points α and β), the distance between α and β is larger than the other distances from point to point. Flipping the common face and constructing edge $\alpha\beta$ reduces to artificially make those points closer while the other are put farthest, [9], and, actually, this operation introduce some degree of anisotropy in the mesh. Indeed, such a flip can be seen as an anisotropic optimization, thus a way to optimize a mesh with respect to an anisotropic metric.

We have then in hand a simple and low cost method which introduces some anisotropy in a given mesh. Nevertheless, the sole use of "2-3" flips results in constructing 3-tet shells which, as well known, are under-connected (there are not enough tets around an edge). Therefore, this sole operator is not fully satisfactory and higher order flips must be envisaged.

Notice, to end the discussion, that constructing anisotropic meshes in this way is an alternate solution

[2]We assume the reader familiar with the difference between a triangulation problem and a meshing problem.

to a direct method (see [8], in french, for such a direct approach).

11. CONCLUSION AND FUTURE WORK

We demonstrated that the natural extension of the "2-2" flip in two dimensions is the edge flips discussed in this paper. We showed it is nothing other than a combination of elementary flips. Various properties of this operator were discussed, including conditions that make a flip possible (thus making a shell of tets reducible). We considered also complexity issues for different purposes (reductibility, optimization, ...). We discussed also about the existence of other edge based operators and a number of applications were envisaged including exotic uses of some degree of anisotropy.

Future works may include computer implementation of anisotropic edge flips (as needed in a general anisotropic mesh generation method), also, a number of applications can be envisaged (as a perspicacious reader can easily imagine !).

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