PLANAR QUADRANGLE QUALITY MEASURES: IS THERE REALLY A CHOICE ?

Philippe P. Pébay[1]

[1] *Sandia National Laboratories, MS 9051 PO Box 969, Livermore CA 94550, U.S.A.*
pppebay@ca.sandia.gov

ABSTRACT

In this article some new results are presented about planar quadrangle quality measures. On the one hand, a modification of the triangle quality measure based on the Frobenius norm is proposed, in order to comply with the case where the reference element is no longer equilateral, but a right isosceles triangle. Provided this modification, a natural use of the Frobenius norm for quadrangular elements is proposed. On the other hand, generalizations of two triangle quality measures, based on edge to inradius ratios, to planar quadrangles are proposed and justified, and their limitations, depending on user's expectations, are exhibited, justifying the use of a more complex quality measure.

Keywords: Mesh generation, quadrangle meshing, mesh quality.

INTRODUCTION

In the context of finite element analysis, it is well known that evaluation of mesh quality is a major issue. In the case of simplicial meshes, a general *concensus* has emerged, based on both experimental and theoretical results, concerning which shape "good" elements should have. As a general solution, one can say without risk that an optimal simplicial element is an equilateral simplex, with respect to a specified metric[1]. *E.g.*, in the particular case of finite element analysis of elliptic problems, [3] shows that accuracy of the approximate solution is directly related to the closeness of such optimal elements.

Several measures have been proposed [1, 7, 11], in order to compute the geometric quality of a given element. More recently, alternative quality measures have been suggested [2, 6, 8], in order to estimate the deviation of a given simplex from the reference optimal element. Unfortunately, few efforts have been made in order to compare and examine the respective merits of the various quality measures that have been proposed. For triangle quality measures, [9, 10] propose an exhaustive study of this issue, while [5, 11] provide some results for tetrahederal meshes.

Concerning planar quadrangles, few general results are available. [12] proposes to estimate the quality of such elements by the means of several criterions, such as *aspect ratio*, *skewness* and *stretching factor*. Albeit natural for some geometries, such measures are not as well defined in a general context. Alternate measures have been proposed by [7], without providing comparisons between them. Therefore, this article examines the extension of some triangle quality measures to quadrangles, and in particular provides a full analyis of the extension of ζ, the semiperimeter to inradius ratio. It is shown that this quality measures satisfies the desired extremal and, depending on the context, asymptotic properties. When these asymptotic properties do not comply with the requirements of the application, the reasons why a quality measure introduced in [7] should be prefered are examined. Concerning the triangle quality measure based on the Frobenius norm and denoted as κ_2 in [9, 10], the nice properties of triangles, in particular the fact that, when they are non-degenerate, their edge vectors are linearly in-

[1] which can be, in particular, anisotropic, *e.g.*, in the context of mesh adaptation.

dependent, do not extend to quadrangles. For this reason, quality measure based on matrix norms cannot be directly extended to such elements. However, since a quadrangle can be decomposed in two different pairs of triangles, one idea is to estimate the quality of these triangles. Hence, this article also examines the efficiency of this idea and proposes an adaptation of matrix norms, when the reference element is no longer an equilateral, but a right isosceles triangle. This allows us to, finally, propose and examine another quality measure for planar quadrangles, based on such matrix norms.

1. ABOUT TRIANGLE QUALITY

1.1 Preliminaries

In this section, we consider a non-degenerate triangle $t = ABC$ with area $\mathcal{A}_t$, semiperimeter p, edges of lengths $a = BC$, $b = AC$ and $c = AB$, and we denote the angle at vertex A (resp. B, C) as α (resp. β, γ) and the radius of the inscribed (resp. circumscribed) circle of t as r (resp. R). In addition, the vertices A, B and C are defined respectively by the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ in an arbitrary orthonormal affine reference frame. For simplicity, we choose a frame of reference parallel to the plane of the triangle t, in which case the coordinates of the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$ and $\mathbf{v}_2$ are respectively denoted as (x_0, y_0), (x_1, y_1) and (x_2, y_2).
The following standard norm-like notations are also used:

$$\begin{aligned} |t|_0 &= \min(a,b,c) \\ |t|_2 &= \sqrt{a^2+b^2+c^2} \\ |t|_\infty &= \max(a,b,c) \\ \theta_0 &= \min(\alpha,\beta,\gamma) \\ \theta_\infty &= \max(\alpha,\beta,\gamma). \end{aligned} \tag{1}$$

Some results from elementary geometry are assumed without proof (see for example [4] for proofs and details). In particular, the following well-known relations will be used:

$$2R = \frac{abc}{2\mathcal{A}_t} = \frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}, \tag{2}$$

where $\mathcal{A}_t$ is given by

$$\mathcal{A}_t = rp, \tag{3}$$

as well as by HERON's formula:

$$\mathcal{A}_t = \sqrt{p(p-a)(p-b)(p-c)}. \tag{4}$$

Finally, it is recalled that the edge ratio is defined as:

$$\tau = \frac{|t|_\infty}{|t|_0}, \tag{5}$$

see [10] for a study of the behavior of τ, with respect to the extremal angles of t.

1.2 Adaptation of κ_2 to right isosceles triangles

An interesting approach to estimate triangle quality has been proposed by various authors (*cf.* [2, 6, 8]), based on the singular values of a matrix which expresses the affine transformation between the mesh element and a given reference element. More precisely, these works have focused on the case where the reference element is a regular simplex, since this element is generally supposed to be the best possible for isotropic simplicial meshes. An in-depth examination of the variations of such quality measures, as well as a comparison with other quality measures has been made in [9, 10].

To our knowledge, the case where the reference element is a right isosceles triangle has not yet been derived; one reason is that such elements do not really correspond to the kind of triangles that are generally wished in the context of finite element analysis. However, in the goal of extending this measure to quadrangular meshes, such elements become naturally the desired ones.

As described in [2], we define an *edge-matrix* of t by:

$$T_0 = \begin{pmatrix} x_1 - x_0 & x_2 - x_0 \\ y_1 - y_0 & y_2 - y_0 \end{pmatrix} \tag{6}$$

and let W be the edge-matrix of a reference isosceles right triangle, for example

$$W = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{7}$$

meaning that W is, simply, the identity matrix of $\mathbb{R}^2$. Hence, $T_0 W^{-1} = T_0$ is the matrix that maps the reference element into t. Using the same ideas as in [2], it is possible to define matrix-norms based on the singular values σ of T_0. Obviously, the symmetry which arises when W is an equilateral triangle vanishes with this new reference element. In particular, t is considered as being optimal only if the right angle is in A. This property allows a strict control, not only over the shape of t, but also on the vertex at which the right angle should be. The singular values are given by the positive square-roots of the eigenvalues of the positive definite matrix $T_0^T T_0$. Now,

$$T_0^T T_0 = \begin{pmatrix} u & w \\ w & v \end{pmatrix}, \tag{8}$$

where

$$\begin{aligned} u &= |\mathbf{v}_1 - \mathbf{v}_0|^2, && (9) \\ v &= |\mathbf{v}_2 - \mathbf{v}_0|^2, && (10) \\ w &= (\mathbf{v}_2 - \mathbf{v}_0)\cdot(\mathbf{v}_1 - \mathbf{v}_0). && (11) \end{aligned}$$

In the above expressions, $\cdot$ denotes the usual scalar product. The singular values σ of T_0 are thus obtained from the characteristic equation of $T_0^T T_0$ as

$$\begin{aligned} \sigma^4 &- \left(|\mathbf{v}_1-\mathbf{v}_0|^2+|\mathbf{v}_2-\mathbf{v}_0|^2\right)\sigma^2 \\ &+ |\mathbf{v}_1-\mathbf{v}_0|^2|\mathbf{v}_2-\mathbf{v}_0|^2 \\ &- \left((\mathbf{v}_2-\mathbf{v}_0)\cdot(\mathbf{v}_1-\mathbf{v}_0)\right)^2 = 0. \end{aligned} \quad (12)$$

Alternatively, this equation can be written as

$$\sigma^4 - 2(b^2+c^2)\sigma^2 + 4\mathcal{A}_t{}^2 = 0. \quad (13)$$

Hence,

$$\sigma_1^2+\sigma_2^2 = b^2+c^2 \quad (14)$$

and $\sigma_1\sigma_2 = 2\mathcal{A}_t$ where σ_1^2 and σ_2^2 $(0<\sigma_1\leq\sigma_2)$ are the two roots of (13). A quality measure can be constructed from the condition number of any unitarily invariant norm of the matrix T_0 (*cf.* [2]). One such family is derived from the SCHATTEN p-norms defined by:

$$N_p(T_0) = (\sigma_1^p+\sigma_2^p)^{1/p} \quad , \quad p\in[1,+\infty[. \quad (15)$$

The case $p=2$ is the FROBENIUS norm, the limiting case $p\to\infty$ is the spectral norm and the case $p=1$ is the trace norm. A non-normalized quality measure is given by the condition number $\kappa_p(T_0)$ which is defined as

$$\kappa_p(T_0) = \left[\left(\sigma_1^p+\sigma_2^p\right)\left(\sigma_1^{-p}+\sigma_2^{-p}\right)\right]^{1/p}. \quad (16)$$

For the particular case $p=2$, and using (2), it follows that

$$\kappa_2(T_0) = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1\sigma_2} = \frac{b^2+c^2}{2\mathcal{A}_t} = \frac{b^2+c^2}{bc\sin\alpha} \quad (17)$$

and, assuming that $\xi=\frac{b}{c}$, which is allowed since t is non degenerate and thus $c\neq 0$, it then follows that

$$\kappa_2(T_0) = \frac{\xi^2+1}{\xi\sin\alpha} = \left(\xi+\frac{1}{\xi}\right)\frac{1}{\sin\alpha} \quad (18)$$

Therefore, consider the mapping

$$\begin{aligned} g:\quad \mathbb{R}_+^*\times]0,\pi[&\longrightarrow \mathbb{R}_+^* \\ (\xi,\alpha) &\longmapsto \left(\xi+\frac{1}{\xi}\right)\frac{1}{\sin\alpha} \end{aligned} \quad (19)$$

which is $\mathcal{C}^\infty$ over the open domain $\mathbb{R}_+^*\times]0,\pi[$; hence, any local *extremum* of g is attained at a stationary point. The first order derivatives are:

$$\begin{aligned} \frac{\partial g}{\partial\xi}(\xi,\alpha) &= \left(1-\frac{1}{\xi^2}\right)\frac{1}{\sin\alpha} \quad (20) \\ \frac{\partial g}{\partial\alpha}(\xi,\alpha) &= -\left(\xi+\frac{1}{\xi}\right)\frac{\cos\alpha}{\sin^2\alpha} \quad (21) \end{aligned}$$

and, given the definition domain, the only stationary point is $(1,\frac{\pi}{2})$. In order to check whether this case corresponds, as expected, to a *minimum*, one has to make sure that the hessian matrix is positive definite. The second-order derivatives are given by:

$$\begin{aligned} \frac{\partial^2 g}{\partial\xi^2}(\xi,\alpha) &= \frac{2}{\xi^3\sin\alpha} \quad (22) \\ \frac{\partial^2 g}{\partial\alpha^2}(\xi,\alpha) &= \left(\xi+\frac{1}{\xi}\right)\frac{\sin^2\alpha+2\cos^2\alpha}{\sin^3\alpha} \quad (23) \\ \frac{\partial^2 g}{\partial\xi\partial\alpha}(\xi,\alpha) &= -\left(1-\frac{1}{\xi^2}\right)\frac{\cos\alpha}{\sin^2\alpha} \quad (24) \end{aligned}$$

which gives, when $(\xi,\alpha)=(1,\frac{\pi}{2})$,

$$\begin{aligned} \frac{\partial^2 g}{\partial\xi^2}\left(1,\frac{\pi}{2}\right) &= 2 \quad (25) \\ \frac{\partial^2 g}{\partial\alpha^2}\left(1,\frac{\pi}{2}\right) &= 2 \quad (26) \\ \frac{\partial^2 g}{\partial\xi\partial\alpha}\left(1,\frac{\pi}{2}\right) &= 0. \quad (27) \end{aligned}$$

Thus, the hessian determinant is equal to $4>0$ and the first diagonal entry is $2>0$. Hence, the hessian matrix is locally positive definite around the critical point, which therefore corresponds to a strict local *minimum* of g. Since g is $\mathcal{C}^\infty$ over its open and connected definition domain, the unicity of the critical point ensures that this *minimum* is, also, absolute. In other words, $\kappa_2(T_0)$ is[2] minimal only for right $(\alpha=\frac{\pi}{2})$ isosceles $(\xi=1\Leftrightarrow b=c)$ triangles. In this case, the value of $\kappa_2(T_0)$ is, obviously 2, which provides the normalization coefficient.

REMARK. It is interesting to examine the case of some usual configurations:

- if t is equilateral, then $\kappa_2(T_0)=\frac{4}{\sqrt{3}}$;
- if t is right in A, then $\kappa_2(T_0)=\frac{b^2+c^2}{bc}$, which equals 2 if and only if $b=c$, and tends to $+\infty$ as either $\frac{b}{c}$ or $\frac{c}{b}$ does;
- if t is right in B, then $\kappa_2(T_0)=\frac{a^2+2c^2}{ac}$, which at best equals $2\sqrt{2}$, when $a=c\sqrt{2}$, and tends to $+\infty$ as either $\frac{a}{c}$ or $\frac{c}{a}$ does. In particular, if t is also isosceles, then $\kappa_2(T_0)=3$.

2. DERIVING QUADRANGLE QUALITY FROM TRIANGLE QUALITY

2.1 Preliminaries

In this section, $q=ABCD$ is supposed to be a non-degenerate convex planar quadrangle, with area $\mathcal{A}$, semiperimeter p, edges of lengths $a=AB$, $b=BC$, $c=CD$ and $d=DA$ and denote the angle at vertex A (resp. B, C, D) as α (resp. β, γ, δ) and θ the arithmetic mean of α and γ. In addition, the vertices A,

[2] with the specified reference element.

B, C and D are defined respectively by the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ in an arbitrary orthonormal affine reference frame. For simplicity, we choose a frame of reference parallel to the plane of quadrangle q, in which case the coordinates of the position vectors $\mathbf{v}_0$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are respectively denoted as (x_0, y_0), (x_1, y_1),(x_2, y_2) and (x_3, y_3). In the particular case of a rectangle, we denote as λ the *stretching factor*, *i.e.* the ratio of the shortest edge to the longest.

We shall also use the following standard norm-like notations:

$$\begin{array}{lcl} |q|_0 & = & \min(a,b,c,d) \\ |q|_2 & = & \sqrt{a^2+b^2+c^2+d^2} \\ |q|_\infty & = & \max(a,b,c,d) \\ \theta_0 & = & \min(\alpha,\beta,\gamma,\delta) \\ \theta_\infty & = & \max(\alpha,\beta,\gamma,\delta). \end{array} \tag{28}$$

Most of the useful metric equalities of triangles do not extend to quadrangles, and this is the first obstacle to the generalization of results such as those presented in [9, 10] in the case of triangles. Nevertheless, HERON's formula can be generalized for quadrangles:

$$\mathcal{A} = \sqrt{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}. \tag{29}$$

This gives an opportunity to discuss the choice of α and γ in the definition of θ. It is well known that the sum of the four angles of a convex quadrangle is equal to 2π; in other words, $\alpha+\gamma$ and $\beta+\delta$ are supplementary thus have opposite cosines, hence equal squared cosines. Therefore, whatever pair of opposite angles is picked in order to define θ, formula (29) returns the same result, *i.e.*, it is symmetrical.

2.2 Quadrangles and triangles

A natural approach to mesure the quality of any given non-degenerate convex quadrangle consists in seeing it as a pair of two non-degenerate triangles sharing one common edge, which is also a diagonal of the quadrangle. Hence, an apparently good idea would be to examine the qualities of these two triangles, but which quality ? Generally speaking (see [9] or [10] for a notable exception), the quality of a triangle is considered to be optimal[3] only for equilateral triangles. Unfortunately, the following example shows that using such triangle quality measurements for quadrangles is not straightforward.

Figure 1 illustrates the case where q is a rhombus, such that one of its diagonals has the same length as its edges. Hence, q can be decomposed either in two equilateral triangles or in two obtuse isosceles triangles. In the general sense of triangle quality, the former case is considered as optimal, while the latter is far from this. In other words, the choice of the particular partition of q in two triangles has an effect over the resulting quadrangle quality measurement; which one shall be chosen ?

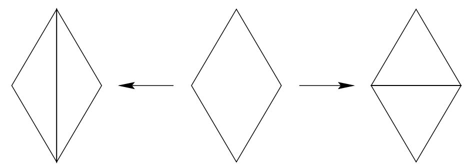

Figure 1: The two triangular covering-ups of the same rhombus.

The only certainty at this point is that either both triangular decompositions of q must be taken into account, or another approach of quadrangle quality, independent from the underlying triangles, must be used.

2.3 The quadrangle space

A quadrangle can be seen as either a geometric or an analytic object. Although the former is certainly more intuitive, the latter allows to use calculus in order to perform an analysis of quadrangle quality measures. A consistent analytic representation of quadrangles is one that is bijective with the set of geometric quadrangles. In addition, quality measures in the general sense (*cf.* [7, 9, 10]) do not depend on size, but only on shape; in other words, they are invariant through homothecy. Hence, it is more suitable to use an analytical representation of quadrangles, up to homothecy.

Equality up to homothecy is an equivalence class, in the strict mathematical sense: reflexive, symmetrical and transitive. Therefore, it allows to define equivalence classes of quadrangles; in particular, any quadrangle q belongs to one and only one equivalence class, and this class is the set of all quadrangles which are homothetic to q. In addition, any equivalence class can be represented by one of its elements, *e.g.*, the only quadrangle with unitary semiperimeter, as illustrated by Figure 2. In other words, considering only the set of such quadrangles is sufficient for quality measure analysis, since these measures must be invariant to scaling. This set, wich is in fact the set of all equivalence classes[4], is simply denoted as Q_1.

Since any edge length of a non-degenerate quadrangle q is strictly smaller than the sum of the three other

[3] more precisely, reaches its strict and unique *minimum*, 1.

[4] or, equivalently, the quotient space of the set of quadrangles by the equality up to homothecy.

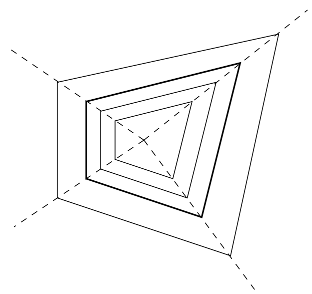

Figure 2: Equality up to homothecy: the unique quadrangle with unitary semiperimeter represents the entire homothecy class.

ones, it follows that

$$\frac{p}{a} = \frac{a+b+c+d}{2a} = \frac{1}{2} + \frac{b+c+d}{2a} > \frac{1}{2} + \frac{a}{2a} = 1 \quad (30)$$

and, for the same reasons, $\frac{p}{b} > 1$, $\frac{p}{c} > 1$ and $\frac{p}{d} > 1$. Hence, denoting as x, y and z the ratios between three edge lengths to the semiperimeter of q, *e.g.*, $x = \frac{a}{p}$, $y = \frac{b}{p}$ and $z = \frac{c}{p}$, it is clear that $x < 1$, $y < 1$ and $z < 1$. In addition,

$$x + y + z = \frac{a+b+c}{p} = \frac{2p-d}{p} = 2 - \frac{d}{p}, \quad (31)$$

whence

$$1 < x + y + z < 2 \quad (32)$$

and

$$0 < 2 - x - y - z = \frac{d}{p} < 1. \quad (33)$$

Hence, on the one hand, the quadrangle q_1 with consecutive edge lengths x, y, z and $2 - x - y - z$ is homothetic to q (with ratio p); on the other hand, its semiperimeter is obviously unitary thus $q_1 \in Q_1$. In other words, q_1 is the class representative of q, on which quality measure analysis shall be performed.

Now, the knowledge of the four edge lengths of a quadrangle is not sufficient to determine its shape[5]. For example, knowing that a quadrangle has four equal edge lengths only allows to conclude that it is a rhombus; nothing is know about the angles of this rhombus which might be, in particular, a square.

Remark. This makes a noticeable difference with the case of triangles, for which there is a bijection between, on the one hand, the ratios between edge lengths and, on the other hand, the angles of this triangle.

In fact, the knowledge of a, b, c, d and, *e.g.*, the angle α, completely determines q and, in particular, the other angles β, γ and δ, for an obvious reason: knowing α allows to determine one of the diagonals, therefore the triangle opposite to α is fully determined by its three edges, according to Remark 2.3. Simply stated, this means that, in addition to edge lengths, quadrangles have one and only one other degree of freedom, provided by any of their four angles. Thus, the set of quadrangles can be seen as a subset of $\mathbb{R}^5$ and, since angles are invariant through homothecy, Q_1 as a subset of $\mathbb{R}^4$. In other words, the quadrangle shape space is four-dimensional.

Provided these results concerning the analytical characterization of quadrangle shape, it is now possible to examine precisely the extremal and asymptotic properties of quadrangle quality measures.

[5] or, equivalently, the knowledge of x, y and z is not sufficient to determine the homothecy equivalence class.

2.4 Edge ratio

It seems natural to extend τ, the edge ratio which has been defined for triangles, to quadrangles. In this case, we have:

$$\tau = \frac{|q|_\infty}{|q|_0} \quad (34)$$

2.4.1 Extremum

By definition, $\tau \geq 1$, with equality if and only if $|q|_\infty = |q|_0$. In other words, τ has a unique *minimum*, 1, which is strict, and reached for, and only for, rhombii. In particular, even a very flattened rhombus, "close" to a degenerate element, is considered as optimal by τ.

2.4.2 Asymptotic behavior

As Q_1 is four-dimensional, it is not as natural to examine the asymptotic behavior, as in the case of triangles. However, it is possible to examine some usual configurations.

First, if q is rectangle, then the edge ratio $\tau_\perp$ becomes equal to λ, the stretching-coefficient, and it is obvious that, as the rectangle stretches, this ratio grows unbounded. In other words, τ does not consider right angles as being a sufficient condition for a good element.

To the contrary, for any element whose edges have close lengths, τ is close to 1. Flattening this element does not change the value of τ, even when one angle tends to 0.

Given both extremal and asymptotic properties of τ, it appears that this extension of a triangle quality measure is, at best, very specific, at worst, unsuitable.

2.5 Edge to inradius

Among triangle qualities that extend naturally to quadraliterals is the comparison of edge lengths with inradius. Of course, any convex quadrangle does not have, in general, an inscribed circle[6] and, hence, it might seem paradoxical to intend to extend such edge to inradius comparisons to quadrangles. However, in the case of a non-degenerate triangle t, it follows from (3) that:

$$\frac{p_t}{r_t} = \frac{p_t^2}{\mathcal{A}_t} \tag{35}$$

where $\mathcal{A}_t$, r_t and p_t respectively denote the area, inradius and semiperimeter of t. Therefore, this quality measurement can be extended directly to q:

$$\zeta = \frac{p^2}{\mathcal{A}}. \tag{36}$$

Similarly, the *aspect-ratio* can be extended to q:

$$\iota = \frac{p|q|_\infty}{\mathcal{A}}, \tag{37}$$

and these two measures are related *via* the following inequality:

$$\zeta = \frac{p(a+b+c+d)}{2\mathcal{A}} \leq \frac{4p|q|_\infty}{2\mathcal{A}} = 2\,\iota \tag{38}$$

with equality if and only if $p = 2|q|_\infty$, *i.e.* if and only if q is a rhombus.

2.5.1 Extremum

Combining (29) and (3) gives:

$$\zeta = \sqrt{\frac{p^4}{(p-a)(p-b)(p-c)(p-d) - abcd\cos^2\theta}} \tag{39}$$

and, as for triangles, it is much more convenient[7] to try to minimize $h = \frac{1}{\zeta^2}$ rather than to maximize ζ. h is, of course, a function of only the five variables a, b, c, p and θ, and can be expressed as follows:

$$h(a,b,c,p,\theta) = \frac{(p-a)(p-b)(p-c)(a+b+c-p) - abc(2p-a-b-c)\cos^2\theta}{p^4} \tag{40}$$

and it is clear that, for any $a \in \mathbb{R}_+^*$, $h(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}, 1, \theta) = h(a,b,c,p,\theta)$. This means that h only depends on four variables, given by the ratios of three edges lengths to the semiperimeter, plus the "torsion" angle θ. In other words, h is invariant through homothecy, as expected, since ζ is non-dimensional. We therefore study the variations of $\tilde{h} : Q_1 \to \mathbb{R}$, where

$$\begin{aligned}\tilde{h}(x,y,z,\theta) = \; & (1-x)(1-y)(1-z)(x+y+z-1) \\ & + xyz(x+y+z-2)\cos^2\theta\end{aligned} \tag{41}$$

[6] In fact, such an incircle exists if and only if the sums of opposite edge lenghts are equal to each other.

[7] and, obviously, equivalent.

whose first-order derivatives are:

$$\begin{aligned}
\frac{\partial\tilde{h}}{\partial x}(x,y,z,\theta) &= (2x+y+z-2) \\
&\quad \times ((1-y)(1-z) + yz\cos^2\theta) \\
\frac{\partial\tilde{h}}{\partial y}(x,y,z,\theta) &= (x+2y+z-2) \\
&\quad \times ((1-x)(1-z) + xz\cos^2\theta) \\
\frac{\partial\tilde{h}}{\partial z}(x,y,z,\theta) &= (x+y+2z-2) \\
&\quad \times ((1-x)(1-y) + xy\cos^2\theta) \\
\frac{\partial\tilde{h}}{\partial \theta}(x,y,z,\theta) &= xyz(x+y+z-2)\sin 2t
\end{aligned}$$

According to (32), $x+y+z < 2$, hence, since none of x, y nor z are null, the stationary-point condition implies that $2\theta \in \pi\mathbb{Z}$ thus, since $2\theta \in]\,0,\pi[$, necessarily $\theta = \frac{\pi}{2}$. Hence, $(x,y,z,\frac{\pi}{2})$ is a stationary-point if and only if

$$\left\{\begin{array}{lcl}(2x+y+z-2)(1-y)(1-z) &=& 0\\ (x+2y+z-2)(1-x)(1-z) &=& 0\\ (x+y+2z-2)(1-x)(1-y) &=& 0\end{array}\right. \tag{42}$$

which is equivalent, since neither x nor y nor z is equal to 1, to

$$\left\{\begin{array}{lcl}2x+y+z &=& 2\\ x+2y+z &=& 2\\ x+y+2z &=& 2\end{array}\right. \tag{43}$$

whose only solution is, clearly, $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Therefore, $\tilde{h}$ has a unique stationary point, when the quadrangle is a square. Now, the hessian matrix of $\tilde{h}$ in $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{\pi}{2})$ is given by (see Appendix for the detailed derivation):

$$H_{\tilde{h}_\square} = -\frac{1}{8}\begin{pmatrix}4&2&2&0\\2&4&2&0\\2&2&4&0\\0&0&0&1\end{pmatrix} \tag{44}$$

and is clearly negative-definite; more precisely, the characteristic polynomial of matrix

$$\begin{pmatrix}4&2&2&0\\2&4&2&0\\2&2&4&0\\0&0&0&1\end{pmatrix} \tag{45}$$

is $(X-1)(X-2)^2(X-8)$, thus the eigenvalues of $H_{\tilde{h}_\square}$ are $-\frac{1}{8}$, $-\frac{1}{4}$ (double) and -1. Hence, the stationary point is a *maximum*, meaning that ζ reaches its only *minimum* for squares; in addition, this *minimum*, denoted as $\zeta_\square$, is strict.

From these extremal property of ζ can be deduced another one for ι: firstly, since (38) is an equality only for rhombii, it is the case for squares, thus $\zeta_\square = 2\iota_\square$. Now, combining the minimization of ζ with (38), leads to:

$$\iota \geq \frac{\zeta}{2} \geq \frac{\zeta_\square}{2} = \iota_\square, \tag{46}$$

showing that, as ζ, ι is minimal for squares.

2.5.2 Asymptotic behavior

In the case where q is a rectangle, ζ becomes:

$$\zeta = \frac{(|q|_0 + |q|_\infty)^2}{4|q|_0|q|_\infty} = \frac{1}{4}\left(\lambda + \frac{1}{\lambda} + 2\right) \quad (47)$$

and is a function of λ, denoted as $\zeta_\perp$. It is clear that $\zeta_\perp$ is strictly increasing over $[1, +\infty[$ (the definition domain of λ) and that $\lim_{\lambda\to+\infty} \zeta_\perp = +\infty$. Moreover,

$$\zeta_\perp(\lambda) \underset{\tau\to+\infty}{\sim} \lambda = \tau_\perp(\lambda) \quad (48)$$

meaning that ζ behaves asymptotically as τ for rectangles.
If q is a rhombus, then its area can be expressed as:

$$\mathcal{A} = a^2 \sin\alpha \quad (49)$$

thus ζ becomes:

$$\zeta_\diamond(\alpha) = \frac{a^2}{4a^2 \sin\alpha} = \frac{1}{4\sin\alpha} \quad (50)$$

which diverges to $+\infty$ at the bounds of $]0, \pi[$, the domain definition of α.

Another interesting asymptotic case occurs when two consecutive edges of q tend towards being aligned, in other words when q gets close to being a triangle t, as shown Figure 3. In this case, called *triangular degeneracy*, it is obvious that q and t share both perimeter and area. Hence, $\zeta(q)$, in the sense of quadrangle quality, tends towards $\zeta(t)$, in the sense of triangle quality.

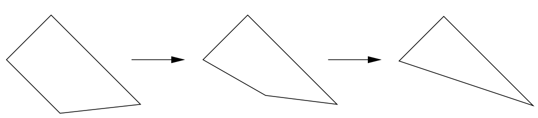

Figure 3: Triangular degeneracy.

From both extremal properties and asymptotic behavior that have been demonstrated, one can conclude that ζ can be extended as a generic quality measure, suitable for both triangles and quadrangles; in addition, a continuous transition between these two kinds of elements is ensured. However, this particular property might be, in some respects, a major drawback since, for some applications, one might wish to avoid triangular degeneracy of quadrangles. In this case, the quality measure should diverge to infinity, rather than tend towards the quality of the limiting triangular element.

2.6 Avoiding triangular degeneracy

[7] propose the following quadrangle quality measure:

$$\mathcal{Q} = \frac{|q|_2\, h_{\max}}{\min_i \mathcal{A}_i} \quad (51)$$

where $\mathcal{A}_i$ denotes the area of the triangle whose edges are those of q adjacent to vertex i and the diagonal opposed to this vertex, and $h_{\max}$ is the *maximum* among $|q|_\infty$ and the two diagonal lengths of q. Obviously, $|q|_\infty \leq h_{\max}$.

First, it is useful to remark that computing each of the $\mathcal{A}_i$ allows to detect, on the fly, whether or not q is convex, non-convex, self-intersected, degenerate (*cf.* [7] for details). Computationally speaking, this is of the greatest interest, since both topological consistency checking and geometrical quality measurement can be done at the same time.

In addition, according to CAUCHY-SCHWARZ inequality,

$$(\forall (u_1, ..., u_n) \in \mathbb{R}_+^n) \quad \sum_{k=1}^{k=n} u_k \leq \sqrt{n \sum_{k=1}^{k=n} u_k^2}, \quad (52)$$

thus $p \leq |q|_2$, with equality if and only if q is a rhombus. Last,

$$\min_i \mathcal{A}_i \leq \frac{\mathcal{A}}{2} \quad (53)$$

with equality if and only if q is a square. Hence,

$$\frac{\zeta}{2} \leq \iota = \frac{p|q|_\infty}{\mathcal{A}} \leq \frac{|q|_2\, h_{\max}}{2\min_i \mathcal{A}_i} = \frac{\mathcal{Q}}{2}. \quad (54)$$

and, in particular, $\zeta \leq \mathcal{Q}$.

An example of how $\mathcal{Q}$ distinguishes elements that neither ζ nor ι would is provided by Figure 4. More precisely, in this case, ABC is a unitary equilateral triangle, while ACD is iscosceles in D, with $AD = CD = x$. Obviously, x must belong to $]\frac{1}{2}, +\infty[$ but, since the

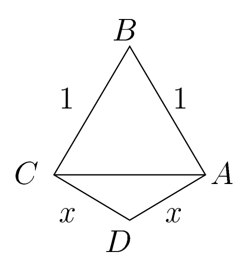

Figure 4: Kite element.

aim is here to examine the case of a triangular degeneracy, the interval is limited to $]\frac{1}{2}, 1[$. It is straightforward to determine ζ, ι and $\mathcal{Q}$ as function of x for

$q = ABCD$. In fact,

$$\min_i \mathcal{A}_i(x) = \frac{1}{4}\sqrt{x^2 - \frac{1}{4}} \quad (55)$$

$$p(x) = 1 + x \quad (56)$$

$$|q|_2(x) = \sqrt{2(1+x^2)} \quad (57)$$

$$|q|_\infty(x) = 1 \quad (58)$$

$$\mathcal{A}(x) = \frac{\sqrt{3}}{4} + \frac{1}{4}\sqrt{x^2 - \frac{1}{4}}. \quad (59)$$

Hence, when $x \to \frac{1}{2}$, both ζ and ι tend towards finite values (respectively, $3\sqrt{3}$ and $2\sqrt{3}$), while $\mathcal{Q} \to +\infty$.

As a matter of fact, $\mathcal{Q}$, as introduced by [7], has all the desired properties for a quadrangle quality measure: extremal and asymptotic. In addition to ζ and ι, it handles triangular degeneracy; in particular, this means that $\mathcal{Q}$ is not continuous with any underlying triangle quality measure.

2.7 Adaptation of κ_2 to quadrangles

It has been shown in Subsection 1.2 how κ_2, as studied in [9] for equilateral triangle, can be adapted when the reference element is a right isosceles triangle with, in addition, a specific control over which edge is the hypotenuse. The main motivation of this modification was to allow, in a second step, to be able to adapt κ_2 to quadrangles.

Considering the generic planar quadrangle q, four different triangles might be evaluated by the means of κ_2: ABD, BCA, CDB and DCA, with respective edge-matrices T_0, T_1, T_2 and T_3. Now, it follows that:

$$T_0 + T_1 + T_2 + T_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (60)$$

i.e.,

$$T_3 = -T_0 - T_1 - T_2. \quad (61)$$

In other words, it is unnecessary to evaluate the four edge-matrices at each vertex of the quadrangle, since any of them is a linear combination of the three other ones. This simply means that, given three vertex angles and edge ratios, the quadrangle is fully determined, up to homothecy.

Now, considering κ_2, as it has been previously modified for right isosceles triangles, the qualities of each of these four triangles are, respectively,

$$\kappa_2(T_0) = \frac{a^2+d^2}{ad\sin\alpha}, \quad \kappa_2(T_1) = \frac{a^2+b^2}{ab\sin\beta}, \quad (62)$$

$$\kappa_2(T_2) = \frac{b^2+c^2}{bc\sin\gamma}, \quad \kappa_2(T_3) = \frac{c^2+d^2}{cd\sin\delta}. \quad (63)$$

According to (60), $\kappa_2(T_3)$ can be directly derived from $\kappa_2(T_0)$, $\kappa_2(T_1)$ and $\kappa_2(T_2)$. However, this dependency is no longer linear, since singular values and, hence, polynomial equations, are involved. Therefore, although it might appear as more elegant to design a quadrangle quality measure, depending only on three of the underlying triangle qualities, it is certainly much more costly. For this reason, a more realistic and certainly more efficient idea is to take into account the four qualities. In this context, a natural approach is to consider their arithmetic mean, and to define the FROBENIUS norm of the quadrangle as follows:

$$\kappa_2(q) = \frac{\kappa_2(T_0) + \kappa_2(T_1) + \kappa_2(T_2) + \kappa_2(T_3)}{4}. \quad (64)$$

REMARK. The choice of the arithmetic mean is an *a priori* without any further justification. One might of course prefer to use the euclidean norm instead. Nevertheless, it would not be a good idea to use a the max norm, as illustrated by Figure 5: $\max_i \kappa_2(T_i)$ cannot detect the fact than one quadrangle is "less" distorted than the other.

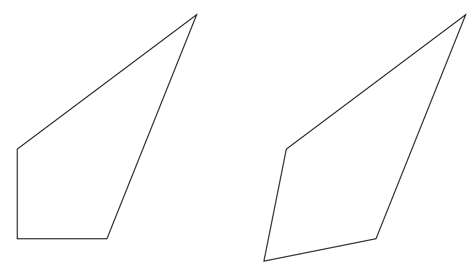

Figure 5: Both quadrangles share the same $\max_i \kappa_2(T_i)$.

2.7.1 Extremum

The previously demonstrated extremal properties of κ_2 for right isosceles triangles show that

$$(\forall i \in \{0,1,2,3\}) \quad \kappa_2(T_i) \geq 2 \quad (65)$$

with equality if and only if T_i is a right isosceles triangle. Hence, $\kappa_2(q) \geq 2$, with equality if and only if each T_i is a right isosceles triangle, *i.e.* q is a square. Hence, $\kappa_2(q)$ complies with the desired extremal property for planar quadrangle quality measures.

2.7.2 Asymptotic behavior

In the case where q is a rectangle,

$$(\forall i \in \{0,1,2,3\}) \quad \kappa_2(T_i) = \frac{a^2+b^2}{ab} \quad (66)$$

from which it follows that

$$\kappa_2(q) = \frac{a^2+b^2}{ab} = \frac{|q|_2^2}{2\mathcal{A}} \quad (67)$$

while if q is a rhombus,

$$(\forall i \in \{0,1,2,3\}) \quad \kappa_2(T_i) = \frac{2a^2}{\mathcal{A}} \qquad (68)$$

whence

$$\kappa_2(q) = \frac{|q|_2^2}{2\mathcal{A}}. \qquad (69)$$

Hence, (67) and (69) appear to be very close of the result that has been exhibited in [9] for κ_2, when the reference element is an equilateral triangle: in this case, the following identity arises:

$$\kappa_2 = \frac{|t|_2^2}{2\sqrt{3}\mathcal{A}_t}. \qquad (70)$$

It is very satisfactory to obtain the same result, up to a constant factor, for, on the one hand, triangles[8], on the other hand rectangles and rhombii. In addition, it also interesting to notice that, in the case where q is a rhombus,

$$\kappa_2(q) = \frac{2}{\sin\alpha} = \frac{2}{\sin\theta} \qquad (71)$$

which gives a direct relation between the quality and the torsion of the rhombus.

Finally, in the case of a triangular degeneracy of q, *i.e.*, at least one of its angles tends to π, it follows immediately from (62) and (63) that at least one of the $\kappa_2(T_i)$ tends to $+\infty$, and so does $\kappa_2(q)$.

CONCLUSION

The results demonstrated in this article concerning quadrangle quality measures can be summarized, as in Table 1. Column "~ 1" indicates which particular element optimizes the normalized quality; column "rectangle λ" provides the asymptotic behavior of the normalized quality when the element is a rectangle with stretching factor λ; column "triangular deg." indicates whether or not the quality measure tends towards infinity in the case of a triangular degeneracy of the quadrangle.

	~ 1	rectangle λ	triang. deg.
τ	rhombus	$\sim \lambda$	no
$\frac{1}{4}\zeta$	square	$\sim \frac{\lambda}{4}$	no
$\frac{1}{2}\iota$	square	$\sim \frac{\lambda}{2}$	no
$\frac{1}{4\sqrt{2}}\mathcal{Q}$	square	$\sim \frac{\lambda}{2}$	yes
$\frac{1}{2}\kappa_2$	square	$\sim \frac{\lambda}{2}$	yes

Table 1: Summary of quadrangle quality measures.

[8]in the usual case, when the reference element is equilateral.

Depending on the specific needs of the user, Table 1 allows to decide which quality measure fits his specific needs, in particular whether divergence to $+\infty$ in the case of a triangular degeneracy is desireable or not.

APPENDIX

The second-order derivatives of $\tilde{h}$, as defined in Paragraph 2.5.1, are given by:

$$\begin{aligned}
\frac{\partial^2\tilde{h}}{\partial x^2}(x,y,z,\theta) &= 2(y+z-yz-1+yz\cos^2\theta)\\
\frac{\partial^2\tilde{h}}{\partial y^2}(x,y,z,\theta) &= 2(x+z-xz-1+xz\cos^2\theta)\\
\frac{\partial^2\tilde{h}}{\partial z^2}(x,y,z,\theta) &= 2(x+y-xy-1+yz\cos^2\theta)\\
\frac{\partial^2\tilde{h}}{\partial \theta^2}(x,y,z,\theta) &= 2xyz(2-x-y-z)\cos 2t\\
\frac{\partial^2\tilde{h}}{\partial x\partial y}(x,y,z,\theta) &= (1-z)(2x+2y+z-3)\\
&\quad + z(2x+2y+z-2)\cos^2\theta\\
\frac{\partial^2\tilde{h}}{\partial x\partial z}(x,y,z,\theta) &= (1-y)(2x+y+2z-3)\\
&\quad + y(2x+y+2z-2)\cos^2\theta\\
\frac{\partial^2\tilde{h}}{\partial y\partial z}(x,y,z,\theta) &= (1-x)(x+2y+2z-3)\\
&\quad + x(x+2y+2z-2)\cos^2\theta\\
\frac{\partial^2\tilde{h}}{\partial x\partial \theta}(x,y,z,\theta) &= yz(2-2x-y-z)\sin 2t\\
\frac{\partial^2\tilde{h}}{\partial y\partial \theta}(x,y,z,\theta) &= xz(2-x-2y-z)\sin 2t\\
\frac{\partial^2\tilde{h}}{\partial z\partial \theta}(x,y,z,\theta) &= xy(2-x-y-2z)\sin 2t
\end{aligned}$$

References

[1] T.J. Baker, Element quality in tetrahedral meshes, *Proc. 7^{th} Int. Conf. Finite Element Methods in Flow Problems*, 1018-1024, Huntsville, U.S.A., 1989.

[2] T.J. Baker, Deformation and quality measures for tetrahedral meshes, *Proc. ECCOMAS 2000*, Barcelona, Spain, September 2000.

[3] P.G. Ciarlet and P.A. Raviart, General Lagrange and Hermite interpolation in $\mathbb{R}^n$ with applications to finite element methods, *Arch. Rational Mech. Anal.* **46**, 177-199, 1972.

[4] H.S.M. Coxeter, Regular Polytopes, MacMillan, New York, 1963.

[5] J. Dompierre *et al.*, Proposal of benchmarks for 3D Unstructured Tetrahedral Mesh Optimization,

Proc. 7th Int. Meshing Roundtable, Sandia National Laboratories, Dearborn, Mi, USA, October 1998.

[6] L.A. FREITAG and P.M. KNUPP, Tetrahedral element shape optimization *via* the Jacobian determinant and condition number, *Proc. 8^{th} Int. Mesh. Roundtable*, South Lake Tahoe, U.S.A., October 1999.

[7] P.J. FREY and P.L. GEORGE, Mesh Generation, Hermes Science Publishing, Oxford & Paris, 2000.

[8] A. LIU and B. JOE, On the shape of tetrahedra from bisection, *Math. Comp.* **63** 207, 141-154, July 1994.

[9] P.P. PÉBAY and T.J. BAKER, Analysis of triangle quality measures, *Math. Comp.*, *to appear.*

[10] P.P. PÉBAY and T.J. BAKER, A comparison of triangle quality measures, *Proc. 10th Int. Meshing Roundtable*, Sandia National Laboratories, Newport Beach, Ca, USA, October 2001.

[11] V.T. PARTHASARATHY, C.M. GRAICHEN and A.F. HATHAWAY, A comparison of tetrahedral quality measures, *Fin. Elem. Anal. Des.* **15**, 255-261, 1993.

[12] J. ROBINSON, Some new distorsion measures for quadrilaterals, *Finite Elements in Analysis and Design* **3**, 183-197, 1987.