Lepp-WCentroid method for tetrahedral mesh improvement

Pedro A. Rodriguez-Moreno\textsuperscript{a,}\textsuperscript{*}, Maria-Cecilia Rivara\textsuperscript{b}

\textsuperscript{a}Universidad del Bio-Bio, Departamento de Sistemas de Información, Av. Collao 1202, Concepción, 4051381, Chile
\textsuperscript{b}Universidad de Chile, Departamento de Ciencias de la Computación, Avenida Beauchef 851, Santiago, 8370456, Chile

Abstract

In this paper we propose an improved Lepp-centroid algorithm for the refinement of 3D triangulations which quickly increases the proportion of better tetrahedra in the mesh. This generalizes and improves previous Lepp-bisection algorithm in three-dimensions. We also study the practical behavior of the algorithm. The algorithm is a simple and fast mesh refinement tool for adaptive finite element methods. This has also great potential to be combined with local mesh improvement operations to quickly eliminate bad quality tetrahedra.

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1. Introduction

In this paper we propose an improved Lepp-WCentroid algorithm for the refinement of 3D triangulations which highly increases the proportion of better tetrahedra in the mesh. The algorithm only performs local refinement operations which maintain conforming meshes throughout the whole refinement process.

It is known that longest edge algorithms for triangulation refinement in 2-dimensions maintain the triangulation quality (smallest angles bounded) while assure that the proportion of better triangles increases as the refinement proceeds \cite{1,4,5}. Even when these properties have not been yet proved in 3-dimensions, practical evidence supports the conjecture that these properties also hold in 3-dimensions. A discussion on a 3D longest edge algorithm and a study on its practical behavior was presented by Rivara and Levin \cite{2}. A different bisection algorithm that assures that a finite number of similar tetrahedra is produced, was discussed (in theory and practice) by Liu and Joe \cite{6}, including a comparison with pure 3D longest edge algorithm.

Later, improved and simpler Lepp based algorithms which completely avoid nonconforming issues were developed \cite{3,4,7–9}. In particular, Lepp bisection algorithm in 3-dimensions, is an efficient reformulation of the longest edge algorithm of reference \cite{2}. This performs longest edge bisections of a set of tetrahedra that share a common longest edge (terminal edge). The Lepp searching path (longest edge propagating path) is used to find the terminal edges associated to each processing tetrahedra.

\textsuperscript{*}Corresponding author. Tel.: +56-41-311-1522; fax: +56-41-311-1040.
E-mail address: prodrigu@ubiobio.cl (Pedro A. Rodriguez-Moreno)
This paper explores a new algorithm that maintains the simplicity of the Lepp bisection algorithm in 3-dimensions by avoiding the bisection patterns that keep some bad tetrahedra in the mesh. This is achieved by selecting a centroid $C_w$ of the set of tetrahedra that share a common longest edge (terminal edge), which is simply inserted in the mesh as a perturbation of the terminal edge bisection operation. The centroid $C_w$ generalizes the terminal triangles centroid selection used in Lepp Delaunay algorithms in 2-dimensions [4,8].

2. Lepp-VCentroid algorithm in 3-dimensions

2.1. Lepp and terminal edge concepts

In two dimensions, Lepp(t), the longest edge propagating path of a triangle $t$ [4,7], is a sequence of increasing triangles that allows to find a unique local largest edge in the mesh (terminal edge) shared by two terminal triangles (one triangle for a boundary terminal edge). For an illustration see Fig. 1 (a). In 3-dimensions Lepp(t) corresponds to a multidirectional searching process that allows to find a set of terminal edges (see Fig. 1 (b)).

Definition $E$ is a terminal edge in a tetrahedral mesh $\tau$ if $E$ is the longest edge of every tetrahedron that shares $E$. In addition we call terminal star $TS(E)$ to the set of tetrahedra that shares a terminal edge $E$.

Definition For any tetrahedron $t_0$ in $\tau$, the Lepp($t_0$) is recursively defined as follows:
(a) Lepp($t_0$) includes every tetrahedron $t$ that shares the longest edge of $t_0$ with $t$, and such that longest edge of $t$ is greater than longest edge of $t_0$.
(b) For any tetrahedron $t_i$ in Lepp($t_0$), this Lepp($t_0$) also contains every tetrahedron $t$ that shares the longest edge of $t_i$ and where the longest edge of $t$ is greater than the longest edge of $t_i$.

Note that Lepp($t_0$) is a 3D submesh with a finite and variable number of associated terminal-edges and terminal stars as schematically shown in Fig. 1 (b).

For each tetrahedron $t$ to be refined, the Lepp-bisection algorithm finds Lepp($t$) and an associated set $W$ of terminal edges. Then for each terminal edge $E$ in $W$, the longest edge bisection of every tetrahedron of the terminal star $TS(E)$ is performed, which is a local refinement operation that maintains a conforming mesh. This procedure is repeated until the target tetrahedron $t$ is refined.

The Lepp WCentroid algorithm selects the WCentroid for point insertion defined as follows:

Serial3DLeppWCentroidAlgorithm($\tau$, $S$)
Input: $\tau$ mesh of tetrahedra; $S$ set of tetrahedra to be refined
Output: refined mesh $\tau_f$
while $S \neq \phi$ do
For each tetrahedron $t_0 \in S$
while $t_0$ remains in the mesh do
Compute Lepp($t_0$) and set of terminal edges $W$
for each terminal edge $L$ in $W$ do
if $L$ is an interior edge then

Fig. 1. (a) Lepp in 2-dimensions: Lepp($t_0$) = {$t_0, t_1, t_2, t_3, t_4$}. AB is the terminal edge; (b) Lepp($t$) in 3-dimensions has several terminal edges $E_i$. 
Compute \( W_{\text{centroid}} C_w \)
Refine terminal star \( TS(L) \) by joining \( C_w \) with the exterior faces of \( TS(L) \)
else
Perform longest edge bisection of the tetrahedra in \( TS(L) \)
end if
end for
end while
end while

**Definition** Let assume we have an interior terminal edge \( E \) with associated terminal star \( TS(E) \) (the set of tetrahedra that share a common longest edge \( E \)). Then we define the \( W_{\text{centroid}} \) of \( TS(E) \) as the average of the centroids of the tetrahedra that belong to \( TS(E) \). Then the following properties hold:

**Proposition 1.** If \( E \) is an interior terminal edge \( E \), then \( C_w \) is in the interior of the terminal star \( TS(E) \).
**Proof.** Each \( C_i \) is inside a tetrahedron \( t_i \) of \( TS(E) \) with interior edge \( E \), which implies that the simple average the \( C_i \) must belong to \( TS(E) \). \( \square \)

The following property supports the elimination of sliver tetrahedra (flat tetrahedra with good triangular faces situated in the equator of its circumsphere), as well as needle, cup and wedge tetrahedra:

**Proposition 2.** (a) Given a sliver tetrahedra \( t_s \), with longest edge equal to a terminal edge. Then the simple insertion of \( C_w \) in the mesh destroys \( t_s \) and produces better tetrahedra. (b) The centroid \( C_i \) of any tetrahedra \( t_i \) in a \( TS(E) \) which is both farther from \( E \) and from the tetrahedra faces, contributes most to the \( C_w \) computation.
**Proof.** Part (a) follows from the fact that the centroid of \( t_s \) is both close to \( E \) and close to the faces of \( t_s \). Consequently the centroids of the other tetrahedra in \( TS(E) \) contribute the most to the \( C_w \) computation. \( \square \)

3. Empirical testing

In three-dimensions, we have used the volume quality measure \( q(t) = CV(t)/L^3 \), where \( V(t) \) is the volume of the tetrahedron \( t \), \( L \) is the length of the longest edge and \( C \) is a constant such that \( q=1 \) for the equilateral tetrahedra which implies \( 0 \leq q(t) \leq 1 \). It is worth noting that every bad quality tetrahedron \( t \) has \( q(t) \) close to 0 for this measure.

We present results for initial non quality meshes of the angel geometry and the spine geometry, respectively (see Figure 2) [11]. To test the improvement properties of the Lepp centroid algorithm and to compare it with Lepp bisection algorithm, we have performed several improvement iterations as follows: given a quality tolerance parameter \( Q_{\text{tol}} \), the first iteration refines all the tetrahedra \( t \) with \( q(t) < Q_{\text{tol}} \). Then the \( i-th \) iteration refines all the non quality tetrahedra remaining in the mesh \( i-1 \), for \( i > 1 \). Figures 3 and 4 allow to visualize the improvement behavior for both algorithms in the 6th iteration for \( Q_{\text{tol}} = 0.2 \).

![Fig. 2. Input meshes.](image-url)
Figures 5 and 6 illustrate the Lepp WCentroid algorithm behavior for different $Q_{tol}$ values for the Angel and Spine geometry, respectively.

![Fig. 3. Angel geometry. Quality distribution comparison: Lepp WCentroid versus Lepp bisection algorithm.](image1)

![Fig. 4. Spine geometry. Quality distribution comparison: Lepp WCentroid versus Lepp bisection algorithm.](image2)

![Fig. 5. Angel geometry. Lepp WCentroid algorithm, quality distribution comparison of iteration 4.](image3)
The Lepp Wcentroid algorithm eliminates the bad quality tetrahedra producing smaller sized meshes as compared with Lepp bisection algorithm. This is very appropriate for adaptive multigrid finite element applications [1]. Furthermore the empirical study performed shows that the algorithm has great potential for fast tetrahedral mesh improvement. For bad quality meshes, we suggest to run some iterations with $Q_{tol} = 0.05$ to eliminate the worst quality tetrahedra combined with standard mesh improvement techniques.

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