Elliptic Fekete points obtained by Ginzburg-Landau PDE

Jovana Jezdimirovića, Alexandre Chemina, Pierre-Alexandre Beauforta,b, Jean-François Remaclea

aUniversité catholique de Louvain, iMMC, Avenue Georges Lemaitre 4, 1348 Louvain-la-Neuve, Belgium
bUniversité de Liège, Montefiore Institute, Allée de la Découverte 10, B-4000 Liège, Belgium

Abstract

The aim of the paper is to present solutions of Smale’s 7th problem i.e. the determination of the minimum potential energy location of \( N \) points \( x_i, \ i = 1, \ldots, N \) on the 2-sphere.

The specific case that is treated here is the Fekete problem where points repel each other with a force that depends on the logarithm of their respective distance. This paper presents a method to compute Fekete points by solving a partial differential equation that is usually used for computing cross fields. This approach is quite uncommon and quite competitive for \( N \) large.

Keywords: Smale’s 7th problem, Fekete problem, minimal energy, partial differential equation.

1. Introduction

In a famous paper [1], Steve Smale details eighteen open mathematical problems for the 21st century. Smale gives there an update for the twenty-three open problems that David Hilbert presented at the 2nd international mathematical congress in 1900.

Smale’s 7th problem is about distribution of points on the 2-sphere \( S^2 \). Consider a set of distinct points \( X_N = \{x_1, x_2, \ldots, x_N\} \) that all belong to \( S^2 \). We use notation \( |x_i - x_j| \) for the Euclidean distance between \( x_i \) and \( x_j \). The energy of point set \( X_N \) is written as:

\[
E(X_N) = - \sum_{i\neq j} \log |x_i - x_j|.
\]

The problem is to find \( X_N \) that minimizes \( E(X_N) \). The optimal points are called elliptic Fekete points. Another way of looking at the problem is to maximize the function:

\[
W(X_N) = \left( \exp E_1(X_N) \right)^{-1} = \left( \prod_{i\neq j} |x_i - x_j|^{-1} \right)^{-1} = \prod_{i\neq j} |x_i - x_j|.
\]

There exists an abundant literature on that problem, both on the theoretical side and on the algorithmic side [1–4]. In this paper, we show that it is possible to find Fekete points by solving a partial differential equation (PDE) on the
sphere. The PDE is essentially the one that is used for computing direction fields in the context of quadrilateral mesh generation [5].

1.1. Ginzburg-Landau PDE

In this section, we aim at showing the profound link between Fekete points and the minimization of the following energy functional, namely Ginzburg-Landau’s functional:

\[ E_\epsilon(u) = \frac{1}{2} \int_G |\nabla u|^2 + \frac{1}{4\epsilon^2} \int_G (|u|^2 - 1)^2. \] (3)

Here, \( u \) represents an element of the quotient space \( S^1/C_N \) where \( S^1 \) is the circle group and \( C_N \) is the group of \( N \)-fold symmetry i.e. rotational symmetry of order \( N \). Computing the first variation of (3) allows to express this minimization problem as the solution of the following PDE:

\[ \nabla^2 u + \frac{1}{\epsilon^2} (1 - |u|^2) u = 0. \] (4)

According to Alberti-Ambrosio-Cabrè theorem for stable solutions of elliptic PDE’s [6], there is a unique minimizer for the given energy \( E_\epsilon(u) \) when \( \epsilon \to 0 \). Moreover, it is also possible to show that [7]:

\[ \lim_{\epsilon \to 0} E_\epsilon = 2N\pi \log(1/\epsilon) + W(x_1, x_2, ..., x_{2N}) + O\left(\frac{1}{|\log \epsilon|}\right). \] (5)

Here, the \( 2N \) points \( \{x_1, x_2, ..., x_{2N}\} \) are critical points i.e. points where \( u \) vanishes. It is indeed possible to show that those points are Fekete points (note here that our approach is only able to compute Fekete points of even orders).

The first term of (5), namely \( 2N\pi \log(1/\epsilon) \) is called singular core energy: energy indeed blows up when \( \epsilon \to 0 \). This term being totally independent of the position of the critical points, it has no influence on their locations. The third term of (5) goes to zero when \( \epsilon \to 0 \), it has no influence as well. The only term of (5) that is finite when \( \epsilon \to 0 \) and that depends on the position of critical points is the second one, \( W \), the renormalized energy:

\[ W(x_1, x_2, ..., x_{2N}) = -\pi \sum_{i \neq j} \log |x_i - x_j|. \] (6)

The position of the \( 2N \) critical points that result of the minimization of the Ginzburg-Landau energy (3) for \( u \in S^1/C_N \) thus correspond to the Fekete points of order \( 2N \).

2. Computational procedure

A finite element method (FEM) has been used to approximate solution of Ginzburg-Landau PDE. A conformal triangulation \( \bigcup \Omega_i = \Omega \) over domain \( \Omega = S^2 \in \mathbb{R}^3 \) is generated. In order to handle \( N \)-folds symmetry issues, a \( N \)-fold symmetry vector field \( u = (u_1, u_2) \):

\[ (u_1, u_2) = (\cos N\alpha, \sin N\alpha), \quad \alpha \in (0, 2\pi), \] (7)

is chosen as the unknown vector. Details on the finite element procedure that has been used can be found in [5]. The principal ingredients are:

- The use of non conforming Crouzeix-Raviart’s triangular finite elements to discretize \( u \),
- A Newton-Raphson’s scheme to solve the non linear system of equations,
- Choice of the coherence length \( \epsilon \) of the order of magnitude of mesh size but way smaller than the distance between critical points.
2.1. Localization of singular points

According to Hairy ball theorem of algebraic topology and following [8], a vector field \( V \) has a singularity at a certain point \( x_i \) if it vanishes or is not defined at this point. Therefore, critical points are detected as the zeros of piecewise linear interpolation of the representation cross-field \( u \) obtained at the triangle element edges of the mesh. In order to localize these critical points on an element size area, elements with the smallest vector norms are extracted. A triangular element, in this sense, is considered to be singular if contains a singular edge. By traversing all existing triangle elements of a mesh in the given domain \( \Omega \), regions with finite number of triangles are obtained. For the purpose of accurate localization of singular points, every region is refined and singular triangular elements with singular points, on the middle of their mutual edges, are extracted. Demonstration of the method (figure 1 and figure 2).

![Fig. 1. Singular points on the mutual edge of two triangle elements.](image1)

![Fig. 2. Geometrical representation of Fekete points for special cases of N - Platonic solids.](image2)

3. Results

According to many authors [1,3,9], there is a variety of computational techniques for obtaining log-optimal configurations: equal area points method [10], spiral points algorithm [3], hexagonal tilings [9], force method approach [11], etc. Drawbacks of above-mentioned algorithms, as restriction for \( N \) large, are crossed over by recent contributions of Cluster Clonetroop, FinisTerrae challenge and MareNostrum numerical experiments [11]. Both theoretical
and computer-aided approaches were mainly relied on geometrical properties of the domain, classic optimization algorithms, combinatorial optimization methods and ODE integrators [11]. Yet, to the best of our knowledge, elliptic Fekete points have never been obtained by a solution of a PDE. Moreover, in comparison with up-to-date calculations for electrostatic potential energy minimum (EPEM) (2), as shown in figure 3, the use of Ginzburg-Landau energy (GLE) (3) is giving competitive results.

![Fig. 3. Preliminary results.](image)

4. Conclusion

This paper presents a contribution in developing a novel approach for generating elliptic Fekete configurations. According to asymptotic behaviour of Ginzburg-Landau functional, the number and localization of singular points obtained by the solution of the cross field $u$ correspond to Smale’s 7th problem. The algorithm used the FEM for solving elliptic Ginzburg-Landau PDE in order to determine these points in an automatic way. As a result, log-optimal configurations, on the unit sphere $S^2 \in \mathbb{R}^3$, were obtained in a robust and efficient way. Depending on requirements, the future work would be addressed to finding reliable solutions for different dimensional cases, topologies and values of $N$ in order to provide more general results on computational cost, convergence and stability of the proposed approach.

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