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Universal Meshes for Domains with Piecewise C^2-regular Boundaries

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Abstract

The idea behind a universal mesh is that a background mesh can be deformed to obtain conforming meshes for a family of domains. Algorithms for universal meshes have so far been limited to geometries with C^2-regular curves. We introduce an algorithm to generate conforming meshes for domains with C^2-regular boundaries. The central idea of the algorithm divides the boundary into a finite number of C^2-continuous curves, and edges of the background mesh are selected for each one of them so that the closest point projection of this set of edges onto the curve is a homeomorphism. We suggest conditions on the background mesh needed to guarantee that the algorithm selects a set of edges that is in fact homeomorphic to the boundary, the proof of which is still in progress.

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Keywords: Universal mesh; FEM; meshing;

1. Introduction

We discuss on a method to generate meshes conforming to a family of domains using a background, initially non-conforming mesh, first introduced in [1]. Because the same background mesh can be deformed to mesh a family of domains, we say that such mesh is a universal mesh for the family. The method perturbs some vertices of a non-conforming mesh to conform to the boundary while maintaining the connectivity of the mesh. This is a trait that the method has in common with quadtree-based methods (e.g. [6]) or isosurface stuffing ([7]). However, a distinguishing feature of the algorithm here is that it works in a very wide class of background, non-conforming meshes, instead of requiring the mesh to adhere to a particular stencil. Insofar, the method may not be the best alternative to generate a specific mesh, but it is a very attractive alternative to robustly obtain meshes when the domain is changing in time, such as in moving domain problems.

Previous studies on universal meshes introduced applications on domains with C^2-regular boundaries, both in $2D$ [2–4] and $3D$ [5]. However, there are still limitations in applications for domains with more general boundaries. We

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describe a method for generating conforming meshes for domains with piecewise C^2-regular boundaries immersed in a non-conforming mesh. The method identifies a group of edges of the background mesh, called positive edges, for each C^2 segment of the piecewise C^2 boundary, and projects the vertices of positive edges onto each segment by the closest point projection to render a conforming mesh. We introduce the conditions that the background mesh has to satisfy to guarantee the robustness of the algorithm, with a qualitative discussion on their roles in the algorithm.

2. Algorithm for Generating Conforming Meshes

Let us consider a domain Ω with a piecewise C^2-regular boundary immersed in a triangulation $\mathcal{T}$. Our objective is to generate a mesh that conforms to $\partial\Omega$ by projecting selected vertices of $\mathcal{T}$ onto $\partial\Omega$, while maintaining the connectivity of $\mathcal{T}$.

2.1. Definitions

The *closest point projection* of $x \in \mathbb{R}^2$ onto a closed set $A \subset \mathbb{R}^2$ is $\pi_A(x) = \arg\min_{y\in A} d(x,y)$, where $d(x,y)$ is the Euclidean distance in $\mathbb{R}^2$. The closest point projection may not be well-defined if x has a number of points in A with a same minimal distance. The *medial axis* of A is the closure of such points in $\mathbb{R}^2$, and is denoted by $\mathcal{M}_A$.

A *simple curve* is a continuous map $C\colon I \to \mathbb{R}^2$ from a closed interval $I = [a,b]$ which is injective in $[a,b)$ and $(a,b]$. In this paper, all curves are simple curves. A curve is *closed* if $C(a) = C(b)$, and is *open* otherwise. A curve is C^2 *continuous* if C is two times differentiable at every point (also $C^{(i)}(a) = C^{(i)}(b)$ for $i = 1,2$ if closed), and *regular* if $\forall s \in I$, $|C'(s)| \neq 0$. It is possible to define $\{\mathbf{t}(s), \mathbf{n}(s)\}$ for a C^2-regular curve, where $\mathbf{t}(s) = C'(s)/|C'(s)|$ and $\mathbf{n}(s)$ is always the 90° rotation of $\mathbf{t}(s)$ either clockwise or counter-clockwise, which determines the *positive side* of C.

For every C^2-regular curve C, there exists $r_C > 0$ such that the map $x(s,\psi)\colon I \times (-r_C, r_C) \to \mathbb{R}^2$ which maps $(s,\psi) \mapsto C(s) + \psi\mathbf{n}(s)$ is a homeomorphism. For such r_C, the domain $N(C) = \{C(s) + \psi\mathbf{n}(s) \mid s \in I, |\psi| < r_C\}$ is called a *tubular neighborhood* of C, and $N^+(C)\colon = \{C(s) + \phi\mathbf{n}(s) \mid s \in I, \phi \in [0,r)\}$ is called the *positive side* of the curve. Then for $x \in N(C)$, we can define the *signed distance of* x to C as $\phi_C(x) = \sigma_C(x) d(x, \pi_C(x))$ where $\sigma_C(x) = +1$ if $x \in N^+(C)$ and -1 otherwise.

A *piecewise C^2-regular curve* $C\colon I \to \mathbb{R}^2$ is a continuous map of an interval $I = [s_0, s_n]$ with a finite number of points $\{s_0, ..., s_n\}$ satisfying $s_0 < s_1 < ... < s_n$, in which $C|_{[s_{i-1},s_i]}$ $(i = 1,2,...,n)$ is a C^2-regular curve. Each $C_i\colon = C|_{[s_{i-1},s_i]}$ is called a *segment*, and each $p_i\colon = C(s_i)$ is called a *corner*. At each corner, we can define the one-sided tangents and one-sided normals as $\mathsf{t}_i^\pm = \lim_{\varepsilon\to\pm 0} \mathbf{t}(s_i + \varepsilon)$ and $\mathsf{n}^\pm = \lim_{\varepsilon\to\pm 0} \mathbf{n}(s_i + \varepsilon)$. The angle of a corner $\angle p_i$ is defined as the (smaller) angle formed by t_i^+ and $-\mathsf{t}_i^-$. In this paper, we assume there is no corner with an angle of 0°.

A *triangulation* or *mesh* $\mathcal{T}$ of an open set $A \subset \mathbb{R}^2$ is a collection of open triangles $\{K_i\}_i$ in $\mathbb{R}^2$ such that: (a) $\overline{A} = \cup_i \overline{K}_i$, (b) $K_i \cap K_j = \emptyset$ if $i \neq j$, and (c) no vertex of any triangle lies in the interior of an edge of another triangle. We say that a domain Ω *is immersed in* $\mathcal{T}$ if $\Omega \subset A$.

Consider a C^2-regular curve C immersed in $\mathcal{T}$. An element K is *positively cut* by C if $\overline{K} \subset N(C)$ and precisely two of its vertices, say v_1, v_2, are in $N^+(C)$. The conditioning angle of K is the angle of vertex $v^* = \arg\min_{v\in\{v_1,v_2\}} \phi_C(v)$, and is denoted by ϑ_K. Either can be chosen if $\phi_C(v_1) = \phi_C(v_2)$. The edge (v_1, v_2) is called a *positive edge* of C, and the union of positive edges is denoted by Γ_C. Fig. 1 illustrates how a positive edge and a conditioning angle are defined.

2.2. Meshing Algorithm

The meshing algorithm is an application of the method for C^2-regular open curves in [4] for each C^2-regular segment, with a judicious selection of the positive side of each segment. It consists of three steps: (a) matching vertices to corners, (b) selecting positive edges, and (c) projecting positive edges. The steps are illustrated in Fig. 2.

In the first step, the nearest vertex v_i^* to each corner p_i is matched to the corner by locally perturbing the mesh. Since moving only the nearest vertex could tangle the mesh, nearby vertices are perturbed to smoothen the mesh. Given a radius $R > 0$, vertices in $B(v_i^*, R)\colon = \{x \in \mathbb{R}^2 \mid d(x, v_i^*) < R\}$ are perturbed using the map

$$v \to v + \left(1 - \frac{d(v_i^*, v)}{R}\right)(p_i - v_i^*), \qquad \forall v \in B(v_i^*, R) \tag{1}$$

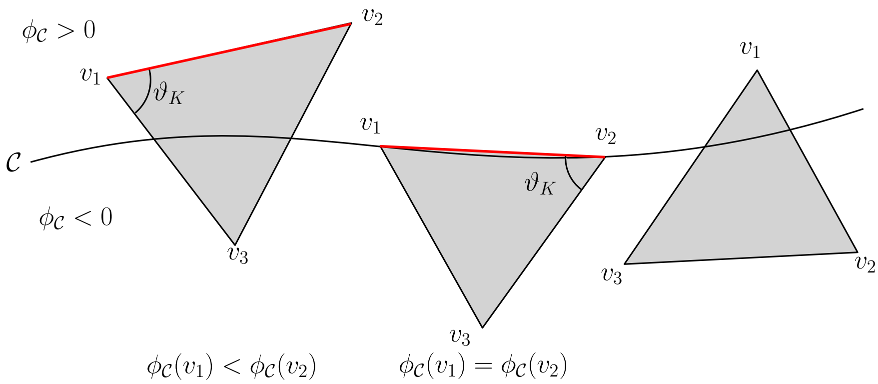

Fig. 1. The first triangle is a positively cut triangle with conditioning angle $\vartheta_K = \angle v_1$. The second triangle is also a positively cut triangle, and either $\angle v_1$ or $\angle v_2$ can be the conditioning angle. The third triangle is not a positively cut triangle.

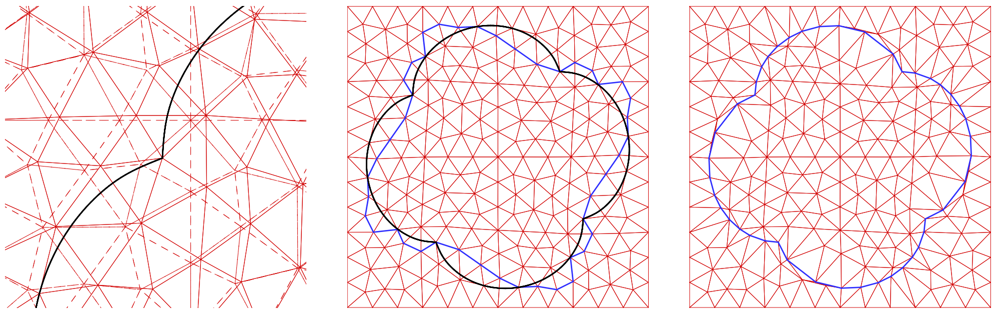

Fig. 2. The illustration shows the steps of generating a conforming mesh. (a) Nearby vertex of the original mesh (dashed) is perturbed to match a vertex to the corner. (b) Positive edges are chosen following the algorithm rule. (c) Positive edges are projected onto the boundary.

where R needs to be sufficiently large relative to the local mesh size.

In the second step, positive edges are defined individually for each segment of the boundary $\partial\Omega$. Positive edges of distinct segments should intersect only at their adjacent corners. For each corner p_i ($i = 1, 2, ..., n$), we choose the positive sides of its neighboring segments to satisfy (a) $\mathsf{n}_i^- \cdot \mathsf{t}_i^+ < 0$ and $\mathsf{n}_i^+ \cdot -\mathsf{t}_i^- < 0$ if $\angle p_i < \pi/2$, and (b) $\mathsf{t}_i^- \cdot \mathsf{n}_i^+ = \mathsf{t}_i^+ \cdot \mathsf{n}_i^-$ if $\angle p_i \geq \pi/2$. If it is not possible to choose the positive sides as above, a segment can be broken into a pair of segments with a corner of angle π. Once the positive sides are defined, positive edges can be identified by the definition. The union of all positive edges $\Gamma_C :\ = \bigcup_i \Gamma_{C_i}$ should be guaranteed to be homeomorphic to C when certain restrictions on the mesh are satisfied.

If there exists a connected subset of positive edges $\gamma \subset \Gamma_{C_i}$ and an edge $e \not\subset C$ such that $\gamma \cup e$ forms a closed curve, then the projection of γ onto Γ_{C_i} may lead to inverted or zero-area triangles. To prevent this failure, such subsets γ are replaced by e, which still guarantees the homeomorphism of closest point projection.

Finally, each set of positive edges Γ_{C_i} ($i = 1, 2, ..., n$) are projected to each segment C_i by the closest point projection. Some vertices near the boundary can be perturbed to relax the mesh and improve its quality. In our examples, we used the mesh relaxation method from [4], which we do not discuss in detail here.

2.3. Conditions on the Mesh

It can be noticed that certain restrictions on the mesh are needed for the algorithm to be successful. There needs to be a bound for the mesh size respect to the curvature, as well as some angles in the mesh. In this note, we focus on the homeomorphism of the projection of positive edges.

Let r_i $(i = 1, 2, ..., n)$ be maximal radius defined for each corner p_i that satisfies

$$\frac{1}{2}(C^{(i)}_{\mathcal{B}_i,r_i} + C^{(i+1)}_{\mathcal{B}_i,r_i})r_i < \cos \max_{K\in\mathcal{B}_i} \vartheta_K, \qquad i = 1, 2, ..., n \tag{2}$$

$$\mathcal{B}_i \cap \mathcal{B}_j = \emptyset, \qquad \forall i \neq j \tag{3}$$

where $\mathcal{B}_i = B(p_i, r_i)$, and $C^{(i)}_{A,d}$ $(i = 1, 2, ..., n)$ is defined for an open set $A \subset \mathbb{R}^2$ and distance $d \in \mathbb{R}$ as

$$C^{(i)}_{A,d} = \frac{M^{(i)}_A}{1 - M^{(i)}_A d}, \qquad M^{(i)}_A = \max_{A\cap C_i} \kappa,$$

We expect that if the mesh size K satisfies the following conditions

$$\overline{K} \subset N(C_i) \qquad \forall \overline{K} \cap int(C_i) \neq \emptyset, \tag{4}$$

$$\frac{1}{2}C^{(i)}_{\pi_{C_i}(K),h_K} h_K < \min\left(\cos\vartheta_K, \sin\frac{\vartheta_K}{2}\right) \qquad \forall \text{positively cut } K, \tag{5}$$

$$\overline{K} \cap \mathcal{M}_C = \emptyset \qquad \forall K \cap B(p_i, r_i) = \emptyset, \tag{6}$$

$$h_K < r_i \qquad \forall K \cap B(p_i, r_i) \neq \emptyset, \tag{7}$$

then the projection $\pi^* : \Gamma_C \to C$ will be a homeomorphism, where $\pi^*(x) = \pi_{C_i}(x)$ for $\forall x \in C_i$ $(i = 1, 2, ..., n)$.

Conditions (4) and (5) are restrictions on the mesh size to guarantee a homeomorphism between each C^2-regular segment and its positive edges. Condition (4) assures that signed distance is well-defined in $\overline{K}$, and (5) restricts the mesh size respect to the local curvature so that the closest point projection of positive edges is a homeomorphism for each segment. It shows a weaker bound compared to [2], but it still requires the conditioning angle ϑ_K to be acute. Conditions (6), (7) are newly introduced bounds for the global homeomorphism of a piecewise C^2-regular curve. Condition (6) assures that positive edges of distinct segments do not intersect away from corners, and (7) gives a bound for the elements near the corner so that the positive edges of adjacent segments intersect only at their corner. We are currently working to confirm condition (7).

3. Examples

We showcase examples of conforming meshes obtained by our algorithm. Fig. 3 shows conforming meshes generated for a family of domains immersed in an identical non-conforming mesh. Each domain consists of 4 C^2-regular segments. Note that it can generate meshes for corners with small angles. For more complex geometries, an adaptive mesh with mesh size small with respect to the curvature of the nearby boundary can be used as a background mesh. Fig. 4 is an example of a conforming mesh for a domain with 11 C^2-regular boundaries, with some extreme curvatures. Notice that the mesh size is larger where the curvature is smaller. An expanded view of the plot is also shown with the quality of the elements. The quality is defined as

$$Q(K) = \frac{\sqrt{3}\triangle K}{4(l_A^2 + l_B^2 + l_C^2)}, \tag{8}$$

where $\triangle K$ is the area of K, and l_A, l_B, l_C are lengths of its sides. The quality is maximum when K is equilateral, and is 0 when an element collapses. All elements show good quality, with minimum quality > 0.448 in the entire mesh.

4. Future Direction

The application of universal meshes on C^2-regular boundaries is yet limited to corners with two adjacent segments. We will proceed to more general cases, such as boundaries with junctions. We also look forward to extend the application to $3D$ domains with edges and corners, such as a cube. Meanwhile, our study on the robustness of our algorithm is limited only in the homeomorphism of the boundary, and it does not show whether the projected mesh can be relaxed to achieve a valid mesh, without any inverted or collapsed elements. We intend to work on the conditions needed for the existence of a relaxed mesh after the projection of positive edges.

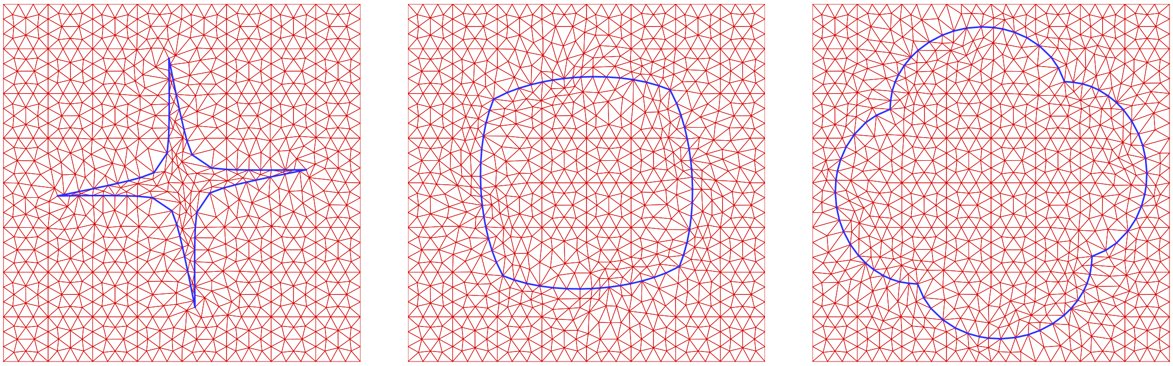

Fig. 3. Conforming boundaries for a family of domains using a universal mesh.

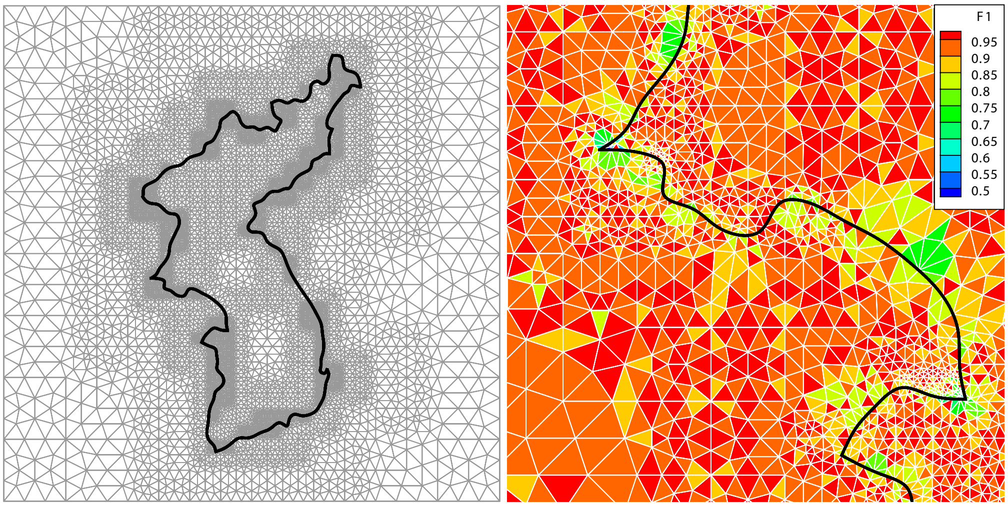

Fig. 4. (Left) Conforming mesh of a map of Korea. The boundary consists of 11 C^2-regular curves. (Right) Expanded view of the boundary with the quality of elements.

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