25TH INTERNATIONAL MESHING ROUNDTABLE

OFF-CENTRES AND RESTRICTED DELAUNAY TESSELLATION:

HIGH-QUALITY MESH GENERATION FOR GENERAL PLANAR, SURFACE AND VOLUMETIC GEOMETRIES

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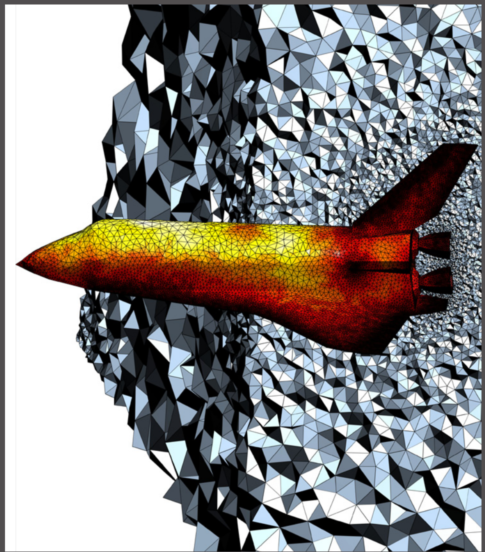
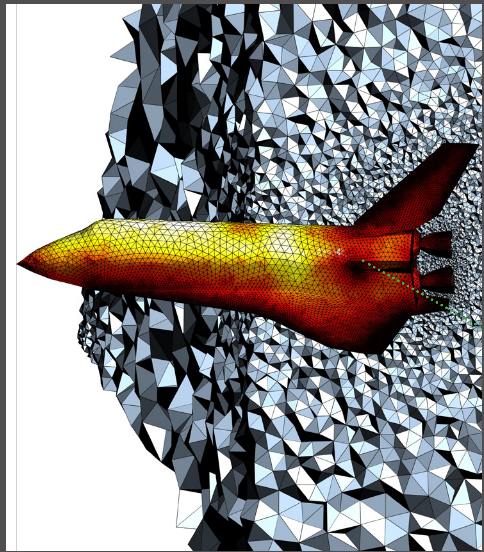
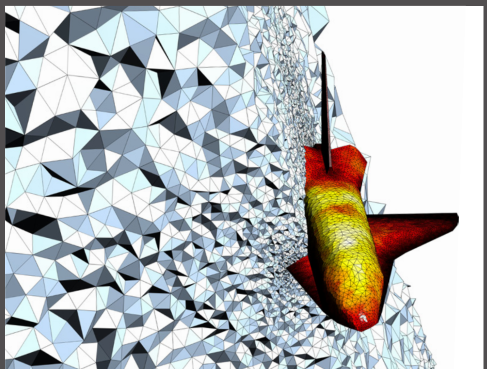
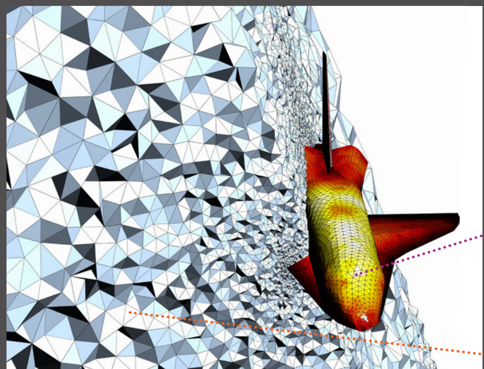


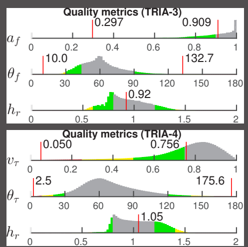


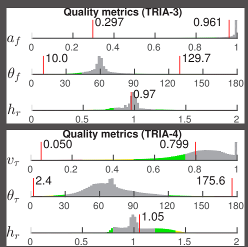




JIGSAW-DR

Meshes for the SHUTTLE geometry using the Delaunay-refinement algorithm, showing normalised histograms of element volume-length ratio, dihedral-angle and relative edge-length. The mesh contains 217,214 surface triangles and 3,644,728 tetrahedrons. Comparing results, it can be seen that the Delaunay-refinement algorithm generates slightly larger meshes with degraded element quality characteristics and mesh-size conformance. Degradation in element volume-length and angle distributions is especially marked in the case of surface triangles, though a similar reductions are also evident in the case of tetrahedral elements. Comparisons of distributions of element relative-length reveal the largest relative differences between algorithms, with the Delaunay-refinement approach displaying significantly weaker conformance to the imposed mesh-size function.





JIGSAW-FD

Meshes for the SHUTTLE geometry using the Frontal-Delaunay algorithm, showing normalised histograms of element volume-length ratio, dihedral-angle and relative edge-length. The mesh contains 191,470 surface triangles and 3,436,473 tetrahedrons. Comparing results, it can be seen that the Frontal-Delaunay algorithm generates slightly smaller meshes with improved element quality characteristics and mesh-size conformance. Improvements in element volume-length and angle distributions is especially marked in the case of surface triangles, though a similar enhancement is also evident in the case of tetrahedral elements. Comparisons of distributions of element relative-length reveal the largest relative differences between algorithms, with the Frontal-Delaunay approach displaying significantly improved conformance to the imposed mesh-size function.

THE JIGSAW MESHING LIBRARY.

JIGSAW is a new Delaunay-based isotropic mesh generation package designed for general two- and three-dimensional meshing problems [1,2,3]. This new environment seeks to combine a number of useful features:

1. A flexible ***geometry-agnostic*** formulation based on ***restricted*** Delaunay tessellations, supporting generalised geometry inputs including piecewise linear complexes, analytic forms and implicit representations.

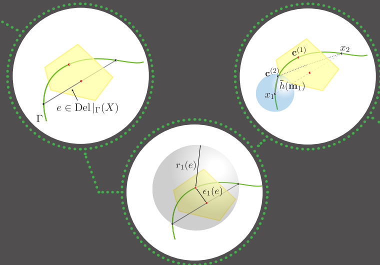
2. A hybrid ***Frontal-Delaunay*** refinement strategy – seeking to achieve very high-quality Delaunay-based mesh generation through use of a hybrid, advancing-front type strategy. Using an appropriate set of ***off-centre*** point-placement rules, this approach aims to combine the best features of classical Delaunay-refinement and advancing-front type techniques, generating high-quality Delaunay meshes that satisfy a set of theoretical bounds and convergence guarantees.

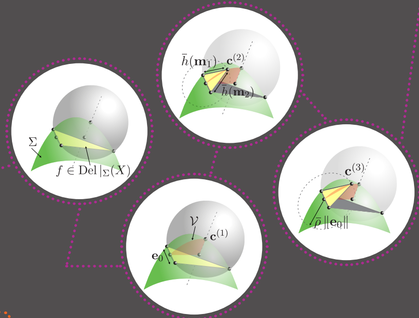
3. A new, publicly-available meshing interface, providing access to a suite of two-and three-dimensional Delaunay-type meshing algorithms via an easy-to-use **MATLAB** / **OCTAVE** scripting interface.

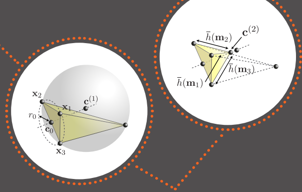
RESTRICTED DELAUNAY TESSELLATION.

Geometrical features are approximated using a hierarchy of Delaunay sub-complexes [4,5,6]. The bounding Delaunay complex DEL(X) contains the restricted sub-complexes $\mathrm{DEL}|_\Gamma(X)$, $\mathrm{DEL}|_\Sigma(X)$ and $\mathrm{DEL}|_\Omega(X)$, providing discrete approximations to the curve network Γ, the surface patches Σ and the interior volume Ω. $\mathrm{DEL}|_\Gamma(X)$, $\mathrm{DEL}|_\Sigma(X)$ and $\mathrm{DEL}|_\Omega(X)$ contain the 1-, 2- and 3-simplexes that best approximate the input curve, surface and volume features, respectively.

Restricted Delaunay techniques exploit the duality between Delaunay tessellations and Voronoi complexes, computing membership for $\mathrm{DEL}|_\Gamma(X)$, $\mathrm{DEL}|_\Sigma(X)$ and $\mathrm{DEL}|_\Omega(X)$ by evaluating intersections between VOR(X) and the input features Γ, Σ and Ω. Such intersections induce a set of circumscribing balls B(Ci,Ri) associated with the faces of DEL(X), providing a set of element-wise refinement points Ci, and a bound on surface discretisation error (Hausdorff distance).







A RESTRICTED DELAUNAY-REFINEMENT ALGORITHM.

Following conventional methodology [4,5,6], a coarse initial Delaunay triangulation is progressively refined until all elements are ***good***. At each step, any that violate a set of constraints are identified and the worst offending elements ***eliminated***. Elimination is achieved through the insertion of additional ***Steiner-vertices*** located at the ***refinement-points*** associated with the elements in question.

```
FUNCTION DELAUNAY-MESH(Γ,Σ,Ω,DEL|Γ(X),DEL|Σ(X),DEL|Ω(X))

   WHILE (BAD(E) or BAD(F) or BAD(T))
      IF (BAD(E)) THEN
         Refine *worst* restricted edge Ei. Insert *refinement-point*
         for Ei and update DEL|Γ(X), DEL|Σ(X), DEL|Ω(X).
      ELSE
      IF (BAD(F)) THEN
         Refine *worst* restricted face Fi. Insert *refinement-point*
         for Fi and update DEL|Γ(X), DEL|Σ(X), DEL|Ω(X). Careful
         not to encroach edges in E.
      ELSE
      IF (BAD(T)) THEN
         Refine *worst* restricted tria Ti. Insert *refinement-point*
         for Ti and update DEL|Γ(X), DEL|Σ(X), DEL|Ω(X). Careful
         not to encroach edges in E or faces in F.
      END IF
  END WHILE

END FUNCTION
```

Elements are considered to be ***bad*** based on a combination of geometrical and topological constraints, including radius-edge ratios, surface-discretisation error measures, element size metrics and local topological considerations.

FRONTAL-DELAUNAY METHODS AND “OFF-CENTRES”.

Frontal-Delaunay algorithms are a hybridisation of advancing-front and Delaunay-refinement techniques, in which a Delaunay triangulation is used to define the topology of a mesh while new vertices are inserted according to an advancing-front type approach.

Standard Delaunay-refinement techniques can be transformed to Frontal-Delaunay type algorithms through the selection of appropriate ***off-centre*** refinement rules [7,1,2,3]:

1A. Edge-refinement: A locally ***size-optimal*** point can be formed by computing intersections between a ball of radius H(Xi) and the curve network Γ. Balls are centred at a frontal vertex. Such a strategy is designed to satisfy local mesh-size constraints.

2A. Face-refinement: A locally ***size-optimal*** point can be found by computing intersections between a disk of radius H(Xm) and the surface patches Σ. The disk is centred on a ***frontal*** edge of the associated face. Such a strategy is designed to satisfy local mesh-size constraints.

2B. Face-refinement: A locally ***shape-optimal*** point can be found by computing intersections between a disk of radius $\rho^* E_{MIN}$ and the surface patches Σ. The disk is centred on a ***frontal*** edge of the associated face. This construction attempts to satisfy local radius-edge constraints.

3A. Tria-refinement: A locally ***size-optimal*** point can be found by computing intersections between a ball of radius H(Xm) and a local Voronoi vector. Such a strategy is designed to satisfy local mesh-size constraints.

3B. Tria-refinement: A locally ***shape-optimal*** point can be found by computing intersections between a ball of radius $\rho^* F_{MIN}$ and a local Voronoi vector. This construction attempts to satisfy local radius-edge constraints.

In each case, it is important to limit off-centre selection to a local ***safe-region*** – an adjacent sub-region of the Voronoi complex bounded by local element circumcentres. This constrained point-selection process preserves the convergence guarantees of the underlying Delaunay-refinement algorithm by ***falling-back*** to standard circumcentre-based refinement in limiting configurations.

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