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Surface reparametrization using quadratic finite elements

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Abstract

The present paper tackles the challenges about one-to-one quadratic finite element mappings, for high order reparametrization purposes. To our best knowledge, it is a first of its kind and this paper enlightens specific problems related to the computation of high order discrete harmonic maps. First, the harmonic mapping from poor quality triangulations is not guaranteed to be bijective and standard convexity arguments do not apply anymore in the high order case. Then, quadratic mappings generate curved parametric triangles that should be valid in order to ensure the correctness of the transformation. Here, we propose a two step procedure that ensures bijectivity: i) curvilinear parametric triangles are generated through a standard finite element procedure and ii) invalid parametric triangles are subsequently untangled. Examples of parameterizations generated with Gmsh are shown.

Keywords: nonstandard geometries; STL triangulations; surface parameterization; harmonic mapping; high order

1. Introduction

Over one million *finite element* (FE) analyses are performed every day in the industry [6]. It is well-known that FE come with the price of mesh generation, which is known to account for about 80% of the whole FE analysis time. The FE method has recently gained interest in new domains such as bio-sciences and geophysics [7]. However, the objects that have to be modelled in those new domains are not defined in the standard fashion. FE mesh generators usually take as input a *computer-aided design* (CAD) model. This model gives a precise blueprint of the objects to be meshed [11]. Actually, the model is given through a *boundary representation* (BREP), which consists of a set of geometrical entities (vertices, edges, faces and volumes) together with their adjacencies. On the other hand, the CAD kernel provides geometrical information of the entities such as parameterizations, curvatures, normals, etc. The geometries defined by a CAD model are said to be *standard*.

In domains such as geophysics and bio-sciences, most of objects are represented in a *nonstandard* fashion. Usually, it is a discrete representation with points cloud, voxels or stereolithography (STL) triangulations. There is a priori no information about the topology, nor the continuous geometry itself. Since there exist procedures to convert points cloud, voxels, etc into triangulations, we focus in this paper on those latter nonstandard geometries. Ordinarily, those triangulations have a very poor quality for numerical simulation purposes. Therefore, there is a real need to enable

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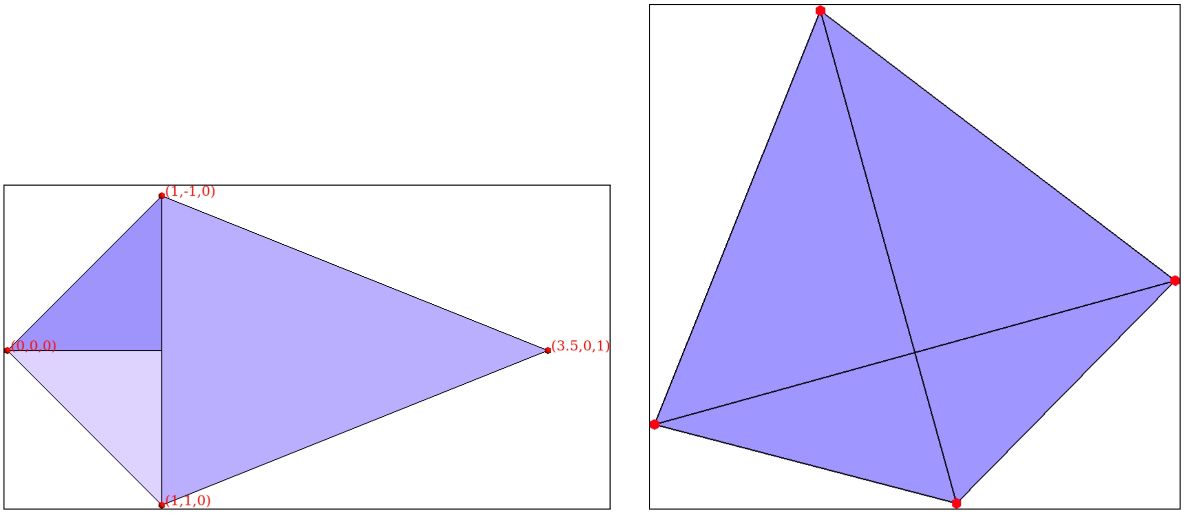


(a) Triangulation $\mathcal{S}_{\mathcal{T}}$. (b) Parametric triangulation within $\hat{\mathcal{S}}$.

Fig. 1. Example of triangulation, which harmonic mapping is not bijective.

mesh generation from discrete geometries, in order to perform high-quality FE analysis on the objects they describe. This can be done by means of a suitable surface parameterization $\mathbf{u}(\mathbf{x})$, i.e. a one-to-one mapping between the surface $\mathcal{S}$ and a 2D parametric domain $\hat{\mathcal{S}}$.

$$\mathbf{x} \in \mathcal{S} \mapsto \mathbf{u}(\mathbf{x}) \in \hat{\mathcal{S}} \tag{1}$$

There are several parameterization methods [1]. Among them, *linear* harmonic mappings are preferred for surface remeshing [4,5] thanks to their ease of computation from a FE formulation of the two Laplace equations. With *quadratic* mappings, parameterizations should be smoother and more suited for meshing.

We begin this paper with a short review about the theory for linear surface parameterization. We underline there the challenge for second (and thus high) order(s) *bijective* harmonic mappings. Afterwards, we present how to get high order harmonic maps, with some results for second order.

2. About linear theory

The linear harmonic mapping usually consists of solving the two following Laplace equations, with appropriate boundary conditions (see [4, equation 11]), thanks to a FE method which mesh is the poor triangulation.

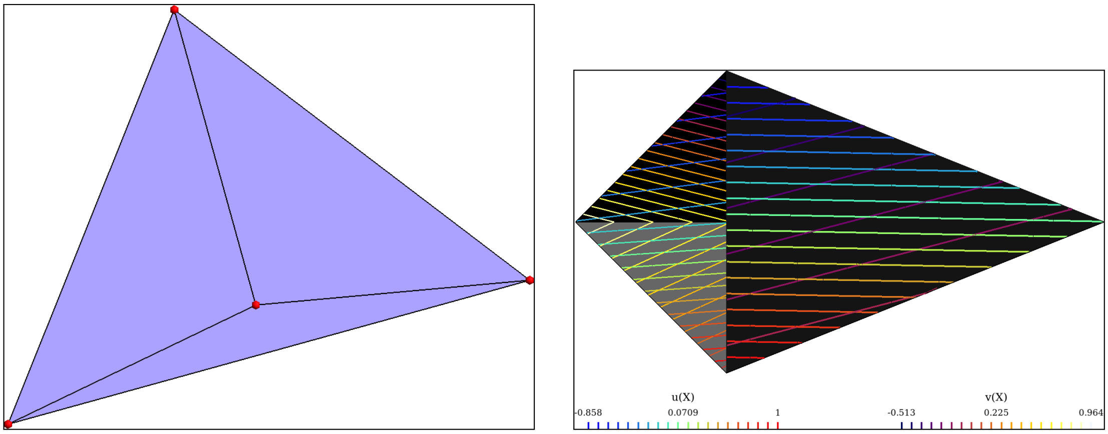
$$\sum_i \frac{\partial^2 u}{\partial x_i^2} = 0, \quad \sum_i \frac{\partial^2 v}{\partial x_i^2} = 0 \tag{2}$$

where $(u; v)$ and $\mathbf{x}$ are respectively *parametric* and *physical* coordinates.

Even if (2) has a unique solution satisfying the *maximum/minimun principle*, it is not ensured for a discrete (weak) solution. Hence, it is possible to get a mapping that is not *injective*, which is not usable. Floater gives an example [8, section 5] which the harmonic mapping is not one-to-one; this is illustrated by figure 1.

It is then relevant to wonder if there exists a piecewise linear bijective mapping for any triangulation. The answer may be given through graph theory, since a triangulation can be seen as a graph: each vertex, edge of a triangle respectively corresponds to a node, edge onto its graph. The problem is thus to determine wether this graph is planar. Tutte [9] proved that those graphs are always planar. Floater [2] showed that it corresponds to have parametric vertices coming from a *convex combination*. This can be expressed as

$$\mathbf{AU} = \mathbf{0} \tag{3}$$



(a) Fixed parametric triangulation within $\hat{\mathcal{S}}$.

(b) Iso-curves $u(\mathbf{x}), v(\mathbf{x})$.

Fig. 2. Floater's example which harmonic mapping is fixed by modifying the discrete system.

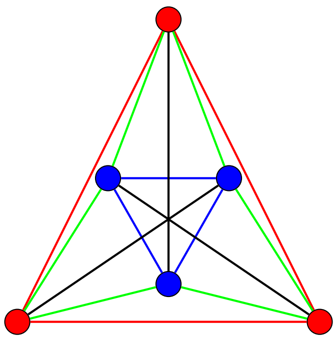


Fig. 3. Graph corresponding to a quadratic triangle: red nodes are the vertices and blue nodes are the high order nodes.

with $\mathbf{A} \in \Re^{n\times n}$ which is a *M-matrix* and $\mathbf{U} \in \Re^{n\times 2}$ which represents parametric coordinates $(u_i; v_i)$ of n vertices.

Such a discrete system arises from the FE formulation of harmonic maps, which can be not bijective; in this case, the matrix A is not a M-matrix. Actually, it can be shown [3, section 3] that the harmonic mapping of a triangulation is ensured to be one-to-one, if for each interior edge, the sum of its opposite angles is less than π. Since meshing is the purpose of our parameterization, we rather modify the discrete system than the triangulation to get a bijective parameterization. Therefore, if we have an harmonic mapping that is not one-to-one, we modify the discrete system to get a M-matrix $\mathbf{A}$.

Unfortunately, for higher orders such as second order, modifying the discrete system to get a M-matrix does not ensure bijectivity of the parameterization. The convex combination underlined by the discrete system is only relevant for *straight* sided parametric triangles, which is not the case for high order parameterization since it produces *curved* parametric triangles. Moreover, the corresponding graph of a quadratic triangle is not planar, as illustrated by figure 3.

3. Towards high order

At first glance, rising the order should not be a big deal, since an harmonic mapping is computed by a FE method. If on one hand it is rather unlikely to get linear harmonic mappings that are not one-to-one, it is likely for second order. The reason is that high order parametrizations produce curved elements in the parametric space $\hat{\mathcal{S}}$. The jacobian of those elements is no more constant, and therefore some elements can be invalid. Besides, usual curved elements

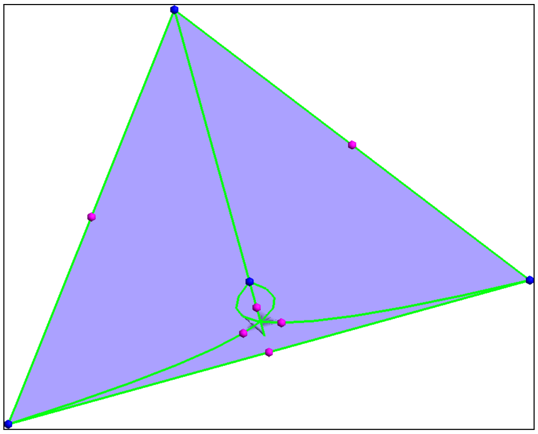
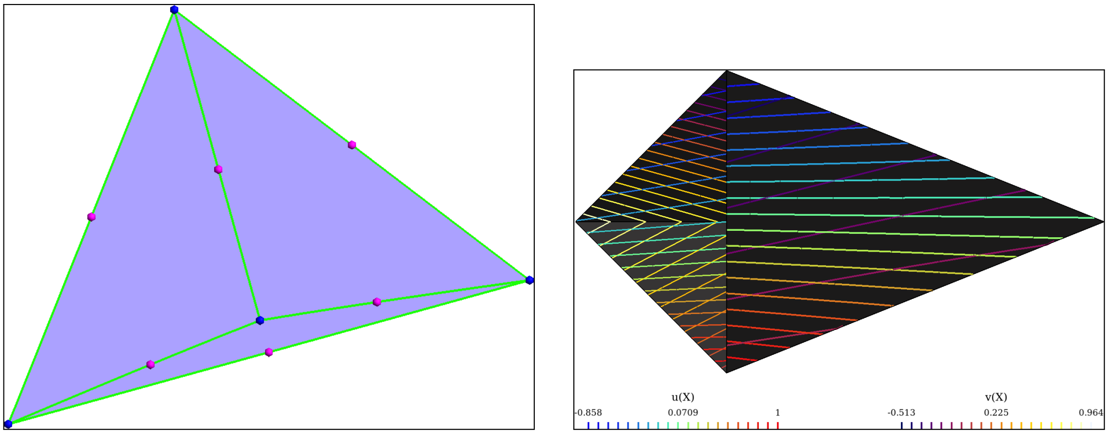


Fig. 4. Floater's example within $\hat{\mathcal{S}}$, computed by a second order harmonic mapping (not one-to-one): blue and purple nodes are respectively the parametric vertices and high order nodes.



(a) Fixed parametric triangulation within $\hat{\mathcal{S}}$.

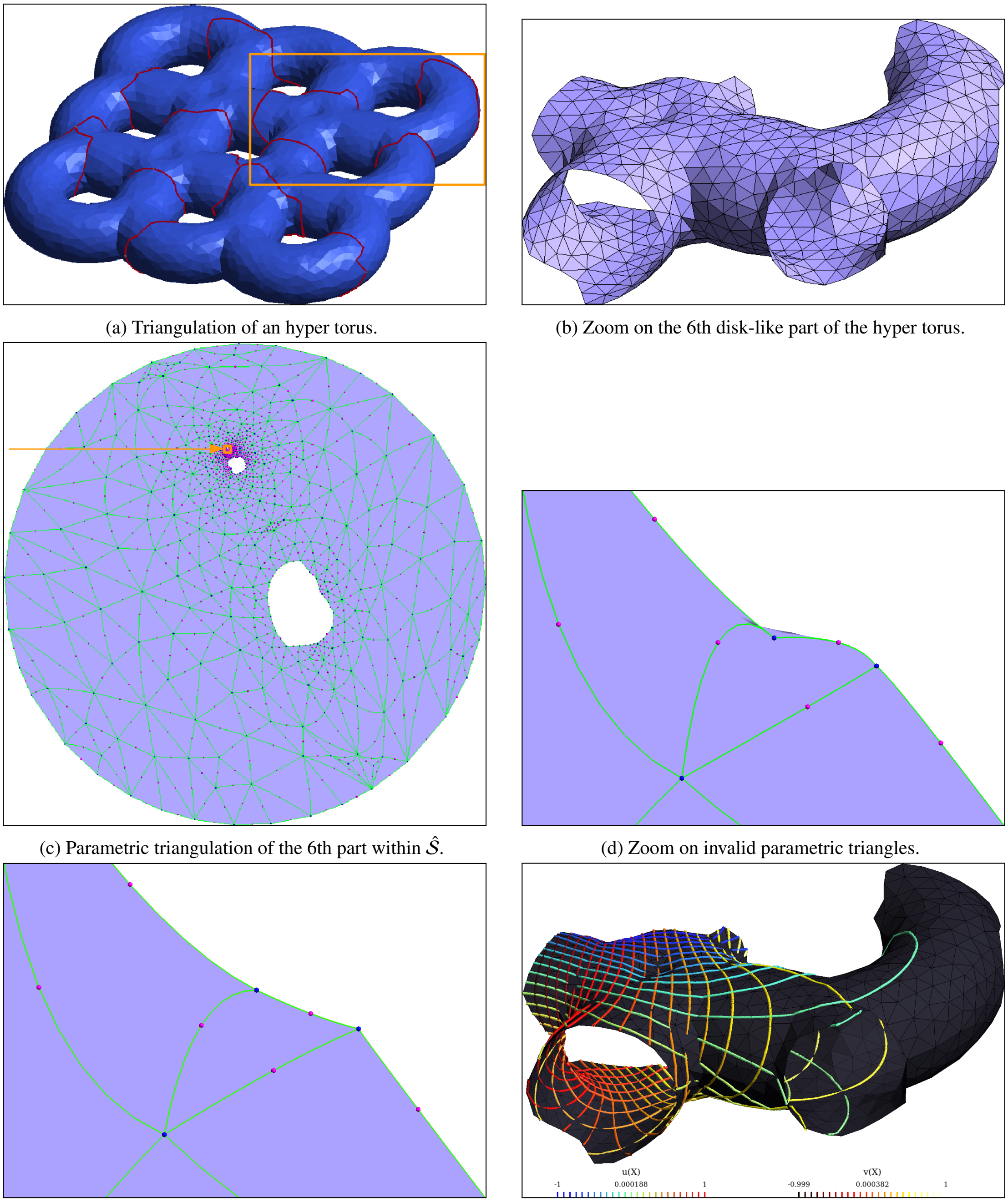
(b) Iso-curves $u(\mathbf{x})$, $v(\mathbf{x})$.

Fig. 5. Floater's example which second order harmonic mapping is fixed by the curve mesh optimizer.

generation needs optimization to get a valid curved mesh - which is not simple. Figure 4 represents the second order harmonic mapping of Floater's example, which is not one-to-one; the jacobian is negative in dark areas and some curved edges intersects in a non (parametric) vertex.

As we say in the previous section, the linear theory is not relevant for high order parameterization. Instead of fixing the discrete system, we fix the parametric mesh (produced by the second order harmonic mapping) with an optimizer of curve mesh [10]. Figure 5 shows the correction for the Floater's example (fig. 4) given by the untangler. We observe that the mapping becomes linear. It is due to the fact that the initial parametric vertices of the quadratic mapping (fig. 4) are close to the ones of the linear mapping (fig. 2 (a)). Since the untangler optimizes on basis of the linear jacobian, it here converges to the linear solution. Let us underline that the untangler does not necessary gives a straight parameterization.

For example, let us parameterize an hyper torus (genus = 9) with our second order harmonic mapping. Since its genus is not zero, it is divided into 7 parts that are disks like (fig. 6 (a)). Some parts do not have bijective parameterization, like the 6th (fig. 6 (b),(c)). It is due to the fact that some parametric triangles are not valid: their jacobian is negative in some area (fig. 6 (d)). Hence, we optimize those triangles to recover them (fig. 6 (e)). Once it is done, we have a one-to-one second order parametrization, illustrated by figure 6 (f).



(a) Triangulation of an hyper torus.

(b) Zoom on the 6th disk-like part of the hyper torus.

(c) Parametric triangulation of the 6th part within $\hat{\mathcal{S}}$.

(d) Zoom on invalid parametric triangles.

(e) Zoom after optimization.

(f) Iso-curves $u(\mathbf{x}), v(\mathbf{x})$ on the 6th disk-like part.

Fig. 6. Hyper torus.

4. Conclusion

We have developed a way to generate high order parameterization, which could enable a new way to represent scattered data with curved triangulation. The challenge was here to get bijective (one-to-one) second order harmonic mapping, from any straight sided triangulation. It is not an easy task, since the tricks of the trade for linear mappings are not relevant for high order parameterizations. It is performed into two steps. First, compute an initial second order parametrization and check that every parametric elements is valid. Then, optimize invalid elements to get stricly positive jacobian, in order to recover a bijective mapping.

Further work will focus on a specific optimizer. At this moment, optimization is based on the linear elements, which is not relevant here. Besides, we will study the parametrization of curve triangulations.

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