A TENTATIVE TO BREAK THE PYRAMID’S CODE

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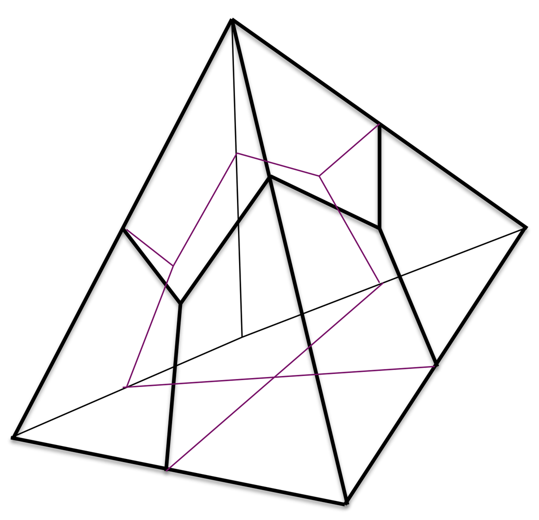
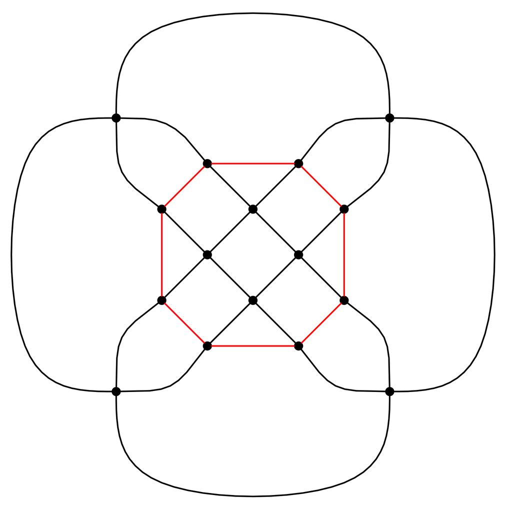
Context

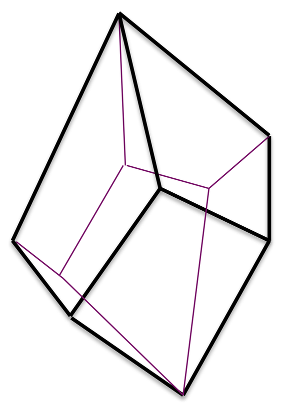
The current minimal solution known to the problem introduced by Robert Schneiders in 1996 [1] is composed of 88 hexahedral elements. This solution was given by Soji Yamakawa and Kenji Shimada in 2010 [2].

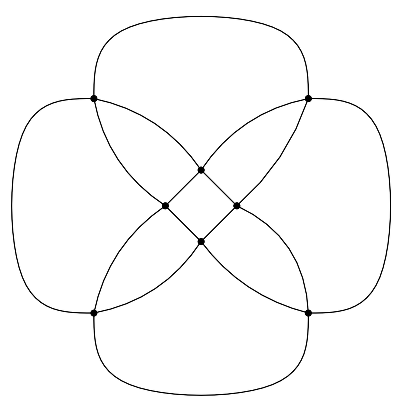




The problem consists in finding an all geometrical hexahedral mesh that conforms to the quadrilateral mesh shown in the left figure. The right side of the figure is the dual graph of the quadrilateral mesh that is made of two chords (one in red, one in black). One of the chord is intersecting itself.

Apparently an even *simpler* problem, is to find the corresponding hexahedral mesh for the 8 quadrilateral mesh shown just below. For this problem, known as the octagonal spindle [3], there is, as far as we know, no solution published yet (even if we suspect that using Jeff Erickson [3] or David Eppstein [4] algorithms it is possible to construct a sub-optimal solution). The dual graph of this quadrilateral mesh is made of only one self-intersecting chord (shown in the right part of the figure).





The question is to find minimal topological hexahedral meshes for those two problems by constructing meshes in a brute force manner: hexahedron after each hexahedron.

To our knowledge, this is the first time that such an attempt is made.

This poster should not encourage people to try to reproduce it !

Methodology

The parallel program named ‘BruteBuilder’ is an attempt to enumerate all the topological hex-meshes composed of *n* elements. The program constructs all the topologies of *n-1* elements and simply add a cube wherever it is possible.

More explicitly the program does at follow:

1. From a configuration **C** of *n-1* cubes, it computes the set of its exterior quadrilateral, noted $\mathbf{Q_C}$
2. Then for each subset **of** $\mathbf{Q_C}$ composed of 1 to 6 exterior quadrilaterals it checks whether it is legal to construct a new topological hexahedron. There is a special case where exactly two opposite faces of the new cube are connected each to the other when it also considers that rotating one of the new quadrilateral (see the results for 3 cubes bellow).

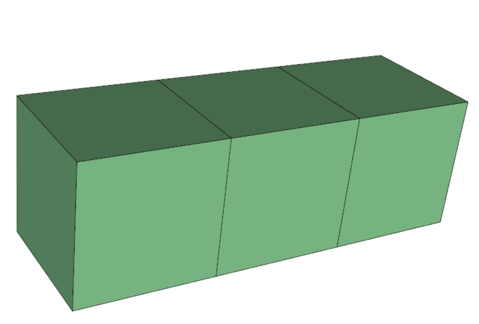
The program excludes some *invalid* topologies so that the intersection of any two topological cubes is either a facet of both, an edge of both, a vertex of both or the empty set.

Methodology (continued)

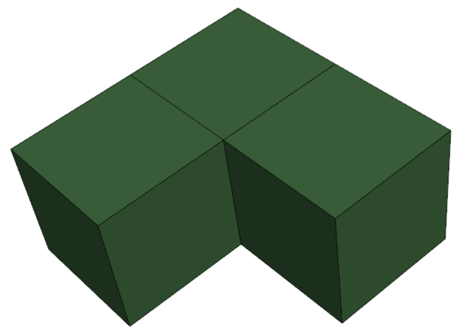
For a low number of cubes – typically up to 7 cubes – the program is able to exclude duplicated topological hex meshes (obtained by topological rotations or symmetries by using an ad-hoc equivalence operator). To do so, topological hex meshes are stored in a hash map indexed by a key based on the dual of its quadrilateral frontier surface : for each quadrilateral it computes a pair of integer that is equals to the number of each of the chords intersecting in the quad. The key is then a summarize of the counts of each kind of pairs. For example the pyramid’s key is 8x[8,16] 8x[16,16], the octagon’s key is 8x[8,8].

There isn’t so much configurations possible with 1 or 2 topological-hexahedra (cubes), and the *interesting part* begins with 3 cubes.

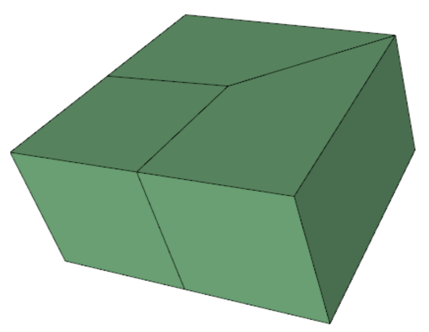
Here our program find 6 different topological configurations, which are geometrically represented below together with their keys.



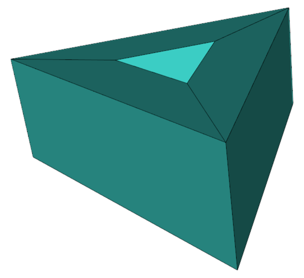
2x[8,8] 12x[4,8]



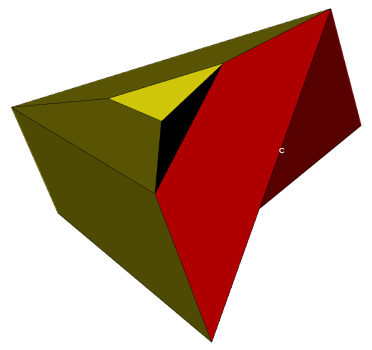
2x[6,6] 4x[4,6] 4x[4,8] 4x[6,8]



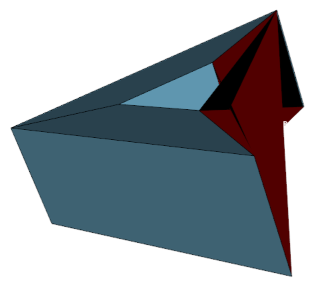
12x[6x6]



12x[3,4]



12x[4,12]



12x[4,6]

The two last configurations clearly show that it is not always easy to find a valid geometrical configuration from valid topological configuration.

Early results

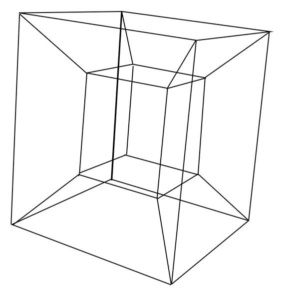
The following table summarizes the numerical results we have.

Number of topological cubes	Number of configurations with respect of rotation and/or symmetries	total # of configurations without respect of…	Minimum number of chords of the surface	Minimum number of faces	# of configuration with <= 3 chords in its surface’s dual
3	6	8	4	12	
4	31	212	4	12	
5	398	1,908	3	12	
6	9,500	34,047	3	10	
7	384,653	1,116,045	2	6	1,103
8	?	58,802,485	0	0	65,012
9	?	12,333,502,269	1	6	<1,736,787
10	?	?	<2	<16	<55,594,090
11	?	?	1	<16	<1,852,347,979

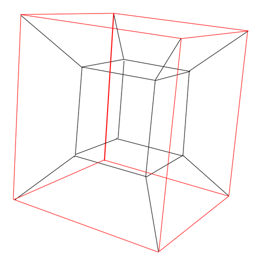
Clearly our program even in parallel cannot enumerate all the configurations with 10 hexahedra. So we chose to filter the configurations with less or than 4 chords in its surface’s dual. This allows us to continue to navigate further in the configurations space.

Most “*interesting*” configurations found so far

- The 2 configurations found with the minimum number of faces:
  - at the left, a well known configuration with 7 topological cubes
  - at the right a 8 topological cubes with no frontier at all ! (The eighth cube is the red one)

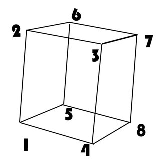


6x[4,4]

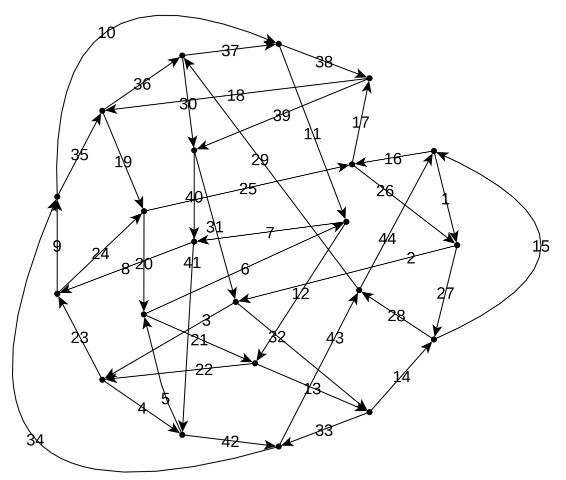


Topologically valid but geometrically invalid !

- A topological configuration with only 1 chord in its frontier composed of 9 hexahedra. It can be constructed with the following hex definition:



H1[0 1 2 3 4 5 6 7] H2[4 5 6 7 8 9 10 11]
H3[0 3 2 1 12 13 14 15] H4[4 8 11 7 13 12 15 14]
H5[0 4 7 3 16 13 14 17] H6 [1 2 6 5 15 14 10 9]
H7[0 1 5 4 16 18 19 13] H8[8 11 15 12 1 18 19 5]
H9[7 14 15 11 3 17 19 18]



Each cube definition is given with the ids of its vertices in the order shown upper. There is 20 different vertices in this configuration and 22 quadrilaterals in its frontier. We really doubt there is a corresponding geometrical configuration! The figure on the right is the dual graph of the quadrilateral frontier mesh.

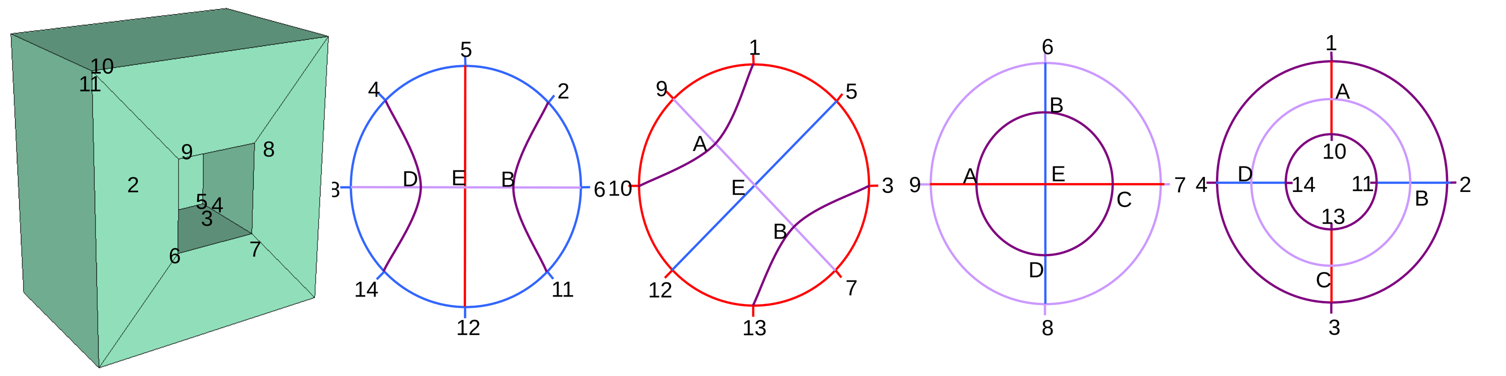
Results and Perspective

Both the Schneiders’ pyramid and the octagonal spindle requires at least 10 hexahedra and probably more than 11 !

The program can be used to find configuration for the simple configuration needed in the “efficient hex-mesh” algorithm [3] for the buffer cubes.

Open problem: from topological to geometrical meshes

Whether from a geometrical hex mesh it is straightforward to derive a topological hex mesh, there are some topological hex meshes that does not have a corresponding geometrical hex mesh. To be more effective in the search of minimum geometric solution for the Schneiders’ pyramid or the octagonal spindle we should use a filter that remove topological meshes that have no geometrical counterpart. If we want to find convex solutions we think that we may put the exterior vertices on a sphere and then try to smooth the interior vertices, but it can be expensive and we do not have solution for non convex geometries like the one shown below (from [5]).



References

[1] Schneiders R. A grid-based Algorithm for the generation of hexahedral element meshes. *Engineering with Computers 1996;* **12**: 168-177.
[2] Yamakawa S. and Shimada K., 88-Element solution to Schneider’s pyramid hex-meshing problem. *International Journal for Numerical Methods in Biomedical Engineering 2010*; **26**:1700-1712.
[3] Erickson, J. Efficiently Hex-Meshing Things with Topology. *Discrete & Computational Geometry 2014*; **52** (3):527-449.
[4] Eppstein D. Linear Complexity hexahedral mesh generation. *Computational Geometry 1999*; **12**: 3-16.
[5] F. Ledoux and J.-Ch. Weill. Extension of the Reliable Whisker Weaving Algorithm. *16th International Meshing Roundtable*, 2007.