
Universal Meshes: Computing Tetrahedralization Conforming to Curved Surfaces.

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1 Introduction

Three dimensional realistic simulation are often unsteady and include moving domains. One of major challenges associated with such simulations is the robust mesh generation for the moving domain. There has been significant research and development in this specific field[1, 2, 3, 4]. We describe a method for generating conforming tetrahedral mesh to the given C^2 continuous surfaces in \mathbb{R}^3 immersed in a non-conforming tetrahedralization. The method consists of constructing a homeomorphic mapping from a subset of tetrahedrons in a background mesh to the ones conforming to the immersed set of surfaces. It relies on the way we parametrize the surfaces of the immersed domains over a collection of a nearby triangular faces with their closest point projections and extension of the same map for the local perturbations of the vertices in the neighborhood these surfaces.

1.1 Universal Meshes

In order to guarantee existence of such a mesh without changing the connectivity of the background mesh, we need to impose restrictions on the background mesh. These restrictions define a family of surfaces for which a conforming tetrahedral mesh can be generated from the background mesh. We say that the background mesh is a universal mesh for such a family of surfaces. The notion of universal meshes is particularly useful in large deformation, fluid solid interaction problems and in numerical schemes that require iterating over the geometry of domains. The same background mesh can serve as the universal mesh for the evolving domains. With no conformity requirements, the universal mesh can be adopted to tetrahedralize large family of domains (C^2 regular) immersed in it, including ones realized over several updates during the course of simulation. This presents a significant algorithmic advantage

for such simulations since this avoids the re-meshing and hosts of issues associated with it. We have presented a way to parameterize the set of C^2 regular surfaces by using the fixed background mesh as a universal mesh in [5].

2 Background Mesh to Conforming Meshes

We consider the problem of generating a conforming mesh using a subset of tetrahedrons from a background mesh \mathcal{T}_h where the selected faces of tetrahedrons conform onto the boundary of a C^2 -regular bounded open domain $\Omega \in \mathbb{R}^3$ while maintaining the connectivity of \mathcal{T}_h .

2.1 Definitions

A *mesh of tetrahedra* in \mathbb{R}^3 is a collection of tetrahedra such that

- each tetrahedron in $T \in \mathcal{T}_h$ is a non-empty set,
- if T_1 and T_2 are distinct tetrahedra in \mathcal{T}_h , then $\overline{T_1} \cap \overline{T_2}$ is either empty, a common face, a common edge, or a common vertex (empty, a common edge, or a common vertex).

The *background mesh* \mathcal{T}_h is a mesh of tetrahedra such that $\partial\Omega$ is *immersed* in \mathcal{T}_h : $\partial\Omega = \overline{\cup_{T \in \mathcal{T}_h} T} \cap \partial\Omega$. A *curved triangle* \tilde{K} is a subset of \mathbb{R}^3 homeomorphic to a triangle. A *mesh of curved triangles* $\tilde{\mathcal{K}}_h$ is a collection of curved triangles defined by a mesh of triangles \mathcal{K}_h and a map $M_h: \Gamma_h \rightarrow \mathbb{R}^3$, with $\Gamma_h = \overline{\cup_{K \in \mathcal{K}_h} K}$, so that $M_h: \Gamma_h \rightarrow M_h(\Gamma_h)$ is a homeomorphism. Each curved triangle $\tilde{K} \in \tilde{\mathcal{K}}_h$ is the image under M_h of a triangle $K \in \mathcal{K}_h$, namely, $\tilde{K} = M_h(K)$. see Fig. 1.

A *mesh of curved triangles over Γ* is a mesh of curved triangles for which $M_h(\Gamma_h) = \Gamma$. We shall alternatively refer to a mesh of curved triangles over Γ as a triangulation of Γ .

We indicate an orientation for Γ with the function

$$s(x) := \begin{cases} -1 & \text{if } x \in \Omega; \\ +1 & \text{otherwise.} \end{cases} \quad (1)$$

Finally, the closest point projection onto Γ , $\pi: \mathbb{R}^3 \rightarrow \Gamma$, is defined as $\pi(x) := \arg \min_{y \in \Gamma} d(x, y)$. Here $d(\cdot, \cdot)$ is the Euclidean distance in \mathbb{R}^3 . Because Γ is a C^2 -regular boundary, $\pi \in C^1(\mathbb{R}^3, \Gamma)$

No conformity is assumed between \mathcal{T}_h and Γ . We next define a mapping M_h and a set \mathcal{K}_h of faces of tetrahedra in \mathcal{T}_h that, under suitable conditions (See [6]), yield a mesh of curved triangles over Γ by deforming the faces in \mathcal{K}_h onto Γ and then extend the same map to generate the conforming tetrahedral mesh \mathcal{T}_h^c .

2.2 Positively cut tetrahedrons and Surface Parameterization

We introduce the terminology of the positively cut tetrahedrons by Γ . We say that a tetrahedron in \mathcal{T}_h is positively cut by Γ if $s = +1$ at three of the four vertices of the tetrahedron and $s = -1$ at one vertex. We call the face shared by the vertices having $s = +1$ in the positively cut tetrahedron, a positive face, \mathcal{K}_h , with respect to Γ . The union of positive faces in \mathcal{T}_h is denoted by Γ_h . As the map to construct the mesh of curved triangles over Γ we set $M_h = \pi|_{\Gamma_h}$, namely, M_h is the restriction to Γ_h of the closest point projection onto Γ . M_h is homeomorphism over any $\mathcal{K}_h \in \Gamma_h$ for sufficiently refined and acute type of tetrahedrons (Theorem proved in [6]). Hence $M_h(\Gamma_h)$ defines a curved triangles mesh of Γ , which is also a parameterization of Γ .

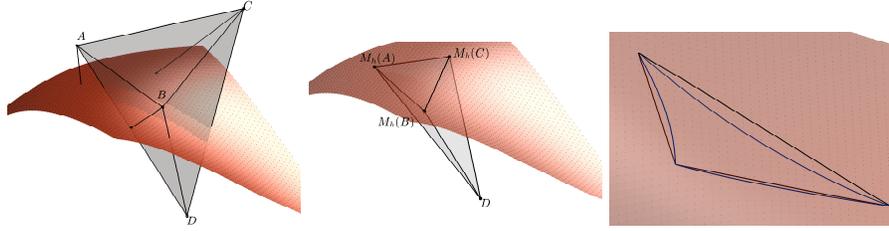


Fig. 1: The figure on the left shows a tetrahedron positively cut by Γ . The arrows shows the corresponding the closest point projection of the positive vertices. The figure in the middle shows the projection of the positive vertices via M_h as well as the corresponding $T \in \mathcal{T}_h^c$ after the mapping. The one on the right shows the corresponding curved triangle exactly conforming to the surface and approximating triangle in blue and black color respectively.

2.3 Description of the Meshing Algorithm

Let \mathcal{T}_h^{sub} be the collection the tetrahedrons in \mathcal{T}_h such that at least one of the vertices of these tetrahedron belong in $B(\Gamma, R_r) := \{x \in \mathbb{R}^3 : d(x, \Gamma) < R_r \ \& \ s(x) = -1\}$. Here R_r is equal to a few multiple of the mesh size h . The meshing algorithm consists of transforming \mathcal{T}_h^{sub} to \mathcal{T}_h^c and is succinctly summarized as the mapping M_h defined over vertices in \mathcal{T}_h^{sub} as

$$M_h(x) = x - f_h(x)\mathbf{n}(\pi(x)) \quad (2)$$

Here, $f_h(x)$ is defined using the parameters $\alpha \in (0, 1]$ and $R_r > h$ as follows;

$$f_h(x) = \begin{cases} \phi(x), & \text{if } x \in \Gamma_h, \\ \alpha h \max\{0, 1 + \frac{\phi(x)}{R_r}\}, & x \in \Omega. \end{cases} \quad (3)$$

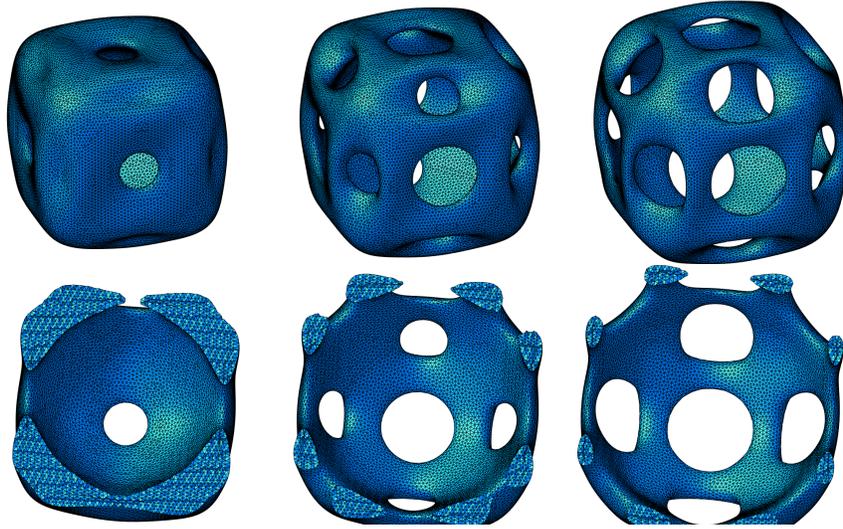


Fig. 2: The conforming tetrahedral meshes and their cut sections for a set of smooth surface generated by using the same background mesh as a universal mesh are shown.

Parameter α is chosen such that $(1 + \frac{h}{R_r})^{-1} \leq \alpha \leq 1$. It is clear that the mapping, M_h , perturbs the vertices along the direction of the local normal to Γ by a distance modulated by f_h (See (2), (3)). The action of M_h on positive faces is: $x \in \Gamma_h \implies f_h(x) = \phi(x) \implies M_h(x) = \pi(x)$. Where the last equality holds when Γ is sufficiently smooth and x lies close to it. In order to accommodate the snapping of vertices of positive faces on to Γ , vertices in the neighborhood $B(\Gamma, R_r)$ are relaxed with respect to Γ based on their distance from Γ . The vertices in \mathcal{T}_h farther than distance R_r from Γ are not perturbed: Since R_r is chosen to be a few multiples of the mesh size, (3) indeed shows that M_h is a local perturbation of vertices near Γ . It is also a small perturbation because the distance by which vertices are perturbed is dictated by f_h and $|f_h| < \alpha h$.

The form of f_h in (3) is particularly suitable for relaxing vertices in background meshes that have uniformly sized tetrahedrons near the boundary. It leaves much to be desired notice that f_h is in fact discontinuous at Γ_h . This is not a problem as we only need control over the magnitude of f_h and its difference quotient over vertices near Γ_h . It would be important to note that it is assumed that the mapping M_h is homeomorphism. We are working towards showing the local homeomorphism of the mapping for any tetrahedra $T \in \mathcal{T}_h^{\text{sub}}$. There is certainly room to improve the choice of f_h , especially to relax vertices by distances as well as choice of directions that lead to overall increase in the quality of the mesh.

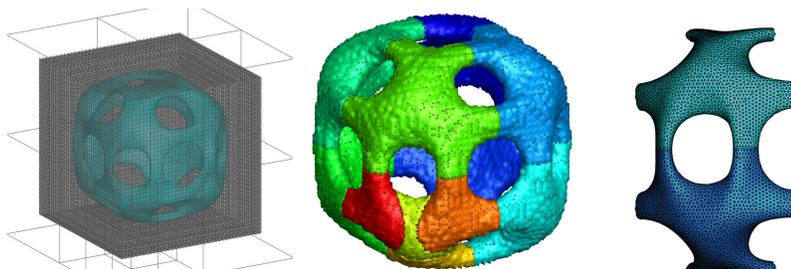


Fig. 3: The figure on the right shows an octree constructed around the given surface and the background mesh. The figure in the middle shows the parts of $\mathcal{T}_h^{\text{sub}}$ to be extracted over different processors in different colors. The last figure shows that the parts of the conforming mesh to be computed over different processors.

3 Examples and Future Direction

We immerse a high genus arbitrarily non-convex C^2 continuous surface in the background tetrahedralization and allow the surface to go through vast topological changes. Fig. 2 show the generated conforming tetrahedral meshes. We intend to thoroughly explore the restrictions required to be imposed on the background tetrahedralization \mathcal{T}_h in order to guarantee the construction of a homeomorphic mapping M_h that leads to a conforming mesh \mathcal{T}_c^h . We plan to pursue this problem in the similar manner as done by Rangaranjan and Lew [1] for universal meshes in \mathbb{R}^2 with non-conforming triangulations. We are working towards an efficient implementation of this algorithm over multiple processors which takes advantage of the explicit mapping (2), a preliminary example is shown in figure 3.

References

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