Universal Meshes for computing with non-conforming tetrahedralization

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1 Introduction

We describe a method for discretizing C^2 continuous surface(s) in $\mathbb{R}^3$ immersed in a non-conforming tetrahedralization. The method consists of constructing a homeomorphic mapping from the tetrahedrons in a background mesh to ones conforming to the immersed geometry. Such a map relies on the way we parametrize the surface(s) of the immersed geometry over a collection of a nearby triangular faces with their closest point projections. In order to guarantee existence of such a parametrization of a surface, we need to impose restrictions on the background mesh. These restrictions define a family of surfaces that can be parametrized with a given background mesh.

1.1 Universal Meshes

We say that the background mesh is a universal mesh for such a family of surfaces. The notion of universal meshes is particularly useful in large deformation problems and in numerical schemes that require iterating over the geometry of domains. The same background mesh can serve as the universal mesh for the evolving domains. With no conformity requirements, the universal mesh can be adopted to triangulate large family of domains immersed in it, including ones realized over several updates during the course of simulating problems with moving boundaries. Hence it facilitates a framework for finite element calculations over evolving domains while using a fixed background mesh. Rangarajan and Lew [1] have proposed universal meshes to achieve conforming mesh for 2D curved domain immersed in non-conforming background triangulation.

2 Background mesh to conforming meshes

Consider a C^2 continuous curved domain Ω that is an open set in $\mathbb{R}^3$ and is immersed in a background tetrahedralization $\mathcal{T}_h$. By Ω *being immersed* in $\mathcal{T}_h$

we mean that the tetrahedralization $\mathcal{T}_h$ contains $\overline{\Omega}$. Note that $\mathcal{T}_h$ is a valid tetrahedralization if

- each tetrahedron in $\mathcal{T}_h$ is a non-empty set.
- if T_1 and T_2 are distinct tetrahedrons in $\mathcal{T}_h$, then $\overline{T}_1 \cap \overline{T}_2$ is either empty, a common face, a common edge or a common vertex.

We take $\Gamma := \partial\Omega$ with an orientation specified as follows.

$$s(x) := \begin{cases} -1 \text{ if } x \in \Omega; \\ +1 \text{ otherwise.} \end{cases} \tag{1}$$

The closest point projection onto Γ, $\pi : \mathbb{R}^3 \to \Gamma$, is defined as follows,

$$\pi(x) := arg \min_{y \in \Gamma} d(x, y) \tag{2}$$

here $d(\cdot,\cdot)$ is the Euclidean distance in $\mathbb{R}^3$. Based on these definitions we can define the signed distance to Γ, $\phi : \mathbb{R}^3 \to \mathbb{R}$ as $\phi(x) := s(\cdot) \times d(\cdot, \Gamma)$. Here $d(\cdot, \Gamma) = \min_{y \in \Gamma} d(\cdot, y)$. At this point we assume no conformity between $\mathcal{T}_h$ and Γ, to be precise no vertex of $\mathcal{T}_h$ needs to lie on Γ. We will define the mapping in the following subsections that would yield a triangulation $\mathcal{T}_h^c$ conforming to Γ by perturbing few vertices of $\mathcal{T}_h$ in the vicinity of Γ.

Positively cut tetrahedrons and positive faces

In order to describe how to perturb the vertices in the background mesh or tetrahedrons $\mathcal{T}_h$, we introduce the terminology of the positively cut tetrahedrons by Γ. We say that a tetrahedron in $\mathcal{T}_h$ is positively cut by Γ if $s = +1$ at three of the four vertices of the tetrahedron and $s = -1$ at one vertex. We call the face shared by the vertices having $s = +1$ in the positively cut tetrahedron, a positive face with respect to Γ. The union of positive faces in $\mathcal{T}_h$ is denoted by Γ_h and the union of positive vertices is denoted by $\mathcal{V}_h^{\Gamma}$.

2.1 Meshing Algorithm

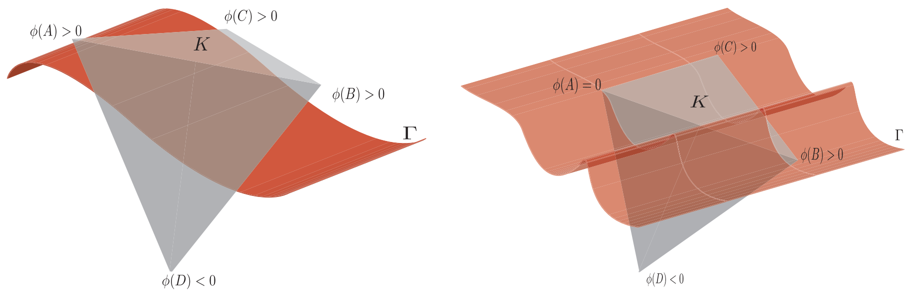
The meshing algorithm consists of transforming Γ_h to $\mathcal{T}_h^c$ and is summarized as the mapping M_h defined over the vertices in Γ_h as follows.

$$M_h(x) = x - \phi(x)\mathbf{n}(\pi(x)) \tag{3}$$

Here $\mathbf{n}$ is the unit outward normal to Γ. It is clear from equation 3 that M_h perturbs the vertices by the signed distance $\phi(x)$ along the direction of the normal to Γ. Hence the action of M_h on the vertices in Γ_h will be as follows,

$$M_h(x) = x - \phi(x)\mathbf{n}(\pi(x)) = \pi(x) \tag{4}$$

The equality holds when x is close to Γ as well as Γ is sufficiently smooth. Hence the vertices in Γ_h are mapped to their closest point projection on Γ.



$\phi(A),\ \phi(B),\ \phi(C) > 0$ and $\phi(D) < 0$. $\phi(A) = 0,\ \phi(B),\ \phi(C) > 0$ and $\phi(D) < 0$.

Fig. 1: Examples to illustrate the definition of a tetrahedron positively cut by Γ and its positive face. In both the cases ABC is the positive face. Note that as shown in 1b it is not necessary that all the points on the positive face ($x \in ABC$) satisfy, $\phi(x) \ngeq 0$ for a tetrahedron to be positively cut by Γ.

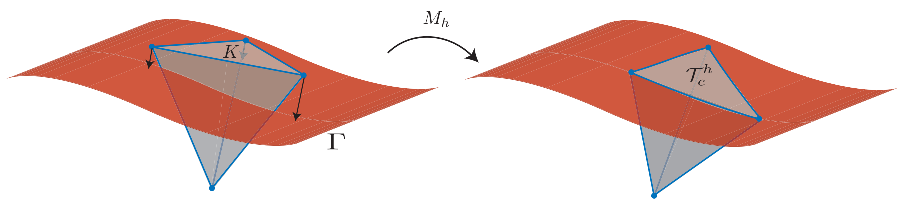


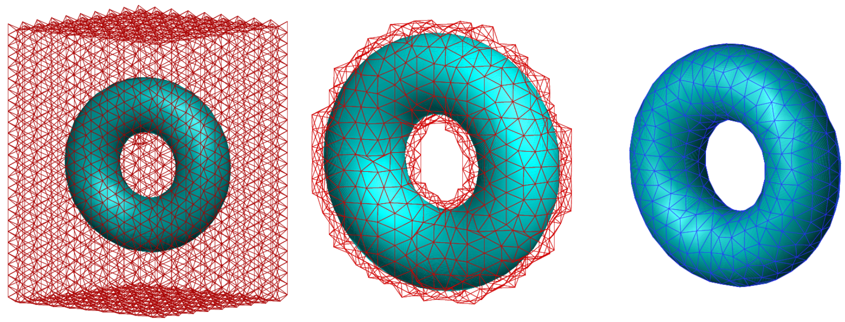
Fig. 2: The figure on the left shows a positively tetrahedron by Γ. We identify positive face and perturb corresponding positive vertices based upon the mapping M_h.

3 Examples

We showcase examples of achieve conforming surface triangulation $\mathcal{T}_c^h$ given a smooth surface Γ. We follow the method described in section 2. Here we have shown examples with surfaces made of bicubic B-spine patch as a building block or a surface given in terms of an implicit equation.

4 Future Direction

We intend to thoroughly explore and understand the restrictions required to be imposed on the background tetrahedralization $\mathcal{T}_h$ in order to guarantee the construction of a homeomorphic mapping M_h to achieve a conforming surface triangulation $\mathcal{T}_c^h$. We plan to pursue this problem in the similar manner as done by Rangaranjan and Lew [2] for universal meshes in 2D with non-conforming triangulations. We also plan to explore the moving boundary problems in 3D based on the scheme proposed by Gawlik and Lew [3].



$\mathcal{T}_h$ Γ_h $\mathcal{T}_c^h$

Fig. 3: Fig. 3a shows the torus immersed in the background mesh $\mathcal{T}_h$. Fig. 3b shows the union of positive faces, Γ_h, in orange wireframe. Fig. 3c shows the conforming triangular mesh $\mathcal{T}_c^h$ for the surface given as a zero level set of a function after applying M_h to Γ_h.

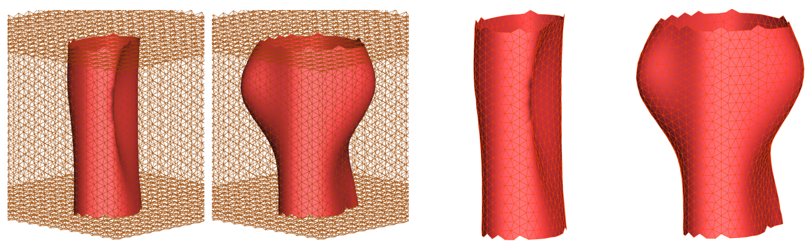


Fig. 4: Figures 4a and 4b shows that all configurations of the moving surface are immersed in the universal mesh. Figures 4c and 4d shows the conforming surface triangulation generated from the universal mesh.

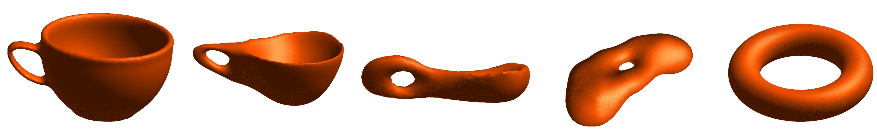


Fig. 5: We show the conforming surface triangulations of various configurations of a coffee cup as it transforms in to a doughnut. All the conforming surface triangulation are generated using the same background mesh $\mathcal{T}_h$ as a universal mesh.

Data: Surface Γ, tetrahedralization $\mathcal{T}_h$.
Result: Conforming triangulation $\mathcal{T}_c^h$.
Identify positive faces
$\Gamma_h \leftarrow \emptyset$;
for all *tetrahedrons* $T \in \mathcal{T}_h$ **do**
 $\{v_a, v_b, v_c, v_d\} \leftarrow$ vertices of T ordered such that
 $s(v_a) \leq s(v_b) \leq s(v_c) \leq s(v_d)$.;
 if $s(v_b) = s(v_c) = s(v_d) = 1$ *and* $s(v_a) = -1$ **then**
  Append the face K of T shared by $\{v_b, v_c, v_d\}$ to Γ_h.;
 end
end
Perturb vertices in Γ_h
for all *faces* $K \in \Gamma_h$ **do**
 Compute π for $\{v_a, v_b, v_c\} \leftarrow$ vertices of K.
 Perturb them based on $M_h(\{v_a, v_b, v_c\}) \in \mathcal{T}_c^h$.
end

Algorithm 1: Achieving conforming triangulation for C^2 regular surface.

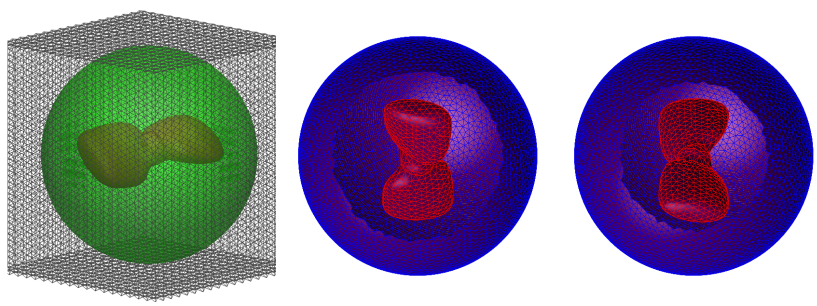


Fig. 6: We have immersed the spherical tank as well as the propeller in the background tetrahedralization $\mathcal{T}_h$. Figures 6b and 6c shows the conforming surface triangulations of various configurations of the propeller as it rotates inside the tank. We can generate conforming surface triangulation for all possible configurations by using the same background mesh $\mathcal{T}_h$ as a universal mesh.

References

1. Ramsharan Rangarajan and Adrian J. Lew. Universal meshes: A new paradigm for computing with nonconforming triangulations. *CoRR*, abs/1201.4903, 2012.
2. Ramsharan Rangarajan and Adrian J. Lew. Analysis of a method to parameterize planar curves immersed in triangulations. *SIAM J. Numerical Analysis*, 51(3):1392–1420, 2013.
3. E. S. Gawlik and Adrian. J. Lew. High-order finite element methods for moving boundary problems with prescribed boundary evolution. *Preprint*, 2013.