Application of Harmonic Mapping on One-To-One Sweeping

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Abstract. The sweeping algorithm can generate all-hexahedral mesh by sweeping an all-quad mesh on the *source* surface to the *target* surface. For one-to-one sweeping, the most difficult thing is to generate an all-quad mesh on the *target* surface which has the same mesh connectivity as that of *source* surface. The traditional method is to use the affine transformations, like translation, rotation, scaling, or combinations of them. This method works very well on the convex cases, while it fails for the concave and non-simply connected cases. In this paper, *harmonic mapping* is used to map an all-quad mesh onto the *target* surface. The result shows that it can generate an all-quad mesh on the *target* surface without any inverted element and avoid expensive smoothing algorithm like untangling.

1 Introduction

In many applications such as *computational fluid dynamics* (*CFD*) [1], hex meshes are preferred over the tetrahedral meshes. According to [2], there are two classes of methods for generating all-hex meshes, namely, indirect methods, which convert from tetrahedral to hexahedral meshes [8], and direct methods. The latter may be further classified as *Grid-Based* [3], *medial surface* [4, 5], *plastering* [6] and *Whisker Weaving* [7]. Because it is difficult to combine or divide the tetrahedron in such a way to guarantee the formation of all-hex mesh, the indirect methods are neither reasonable nor tractable for the mesh generation [2]. For the *Grid-Based* method, mesh quality at the boundary of volume is very poor and interior hex elements are not aligned with boundary hex elements, which don't qualify the ideal requirements for *CFD*. The *medial surface* generates the hex meshes by decomposing the volume, which is an extension of *medial axis* method. The decomposed volumes are usually meshed with midside subdivision. However, this only works for the geometry with 3-valent corner vertices, and lacks reliability for the general geometry. *Plastering* is a 3D extension of the paving algorithm. *Whisker Weaving* method builds the dual of the hexahedral mesh and embeds the mesh in 3D. Neither *plastering* nor *Whisker Weaving* has been shown to be robust for the general 3D models.

While all-hexahedral mesh generation on the general 3D geometry remains an elusive goal, algorithms to mesh two-and-one-half dimensional geometries, generally referred to as "sweeping" or "projection" methods, continue to be important [10, 13]. The traditional one-to-one sweeping procedure consists of four steps: (1) generate an all-quad mesh on the *source* surface; (2) project an all-quad mesh from *source* surface to the *target* surface; (3) generate the structured all-quad meshes on the *linking* surfaces; (4) generate an all-hex mesh, including interior nodes and elements. The *source* and *target* surfaces may have different shapes, areas or curvatures, but they must be topologically equivalent. Of the above four steps, the most difficult one is to map of *source* surface mesh to the *target* surface. This is especially true for models with concave or non-simply connected features.

2 Previous Work.

P. Knupp [10] devised two new algorithms to locate the interior nodes: linear transformations between the bounding node loops and smoothing. For the concave and non-simply connected *source* or *target* surfaces, this approach does not work. X. Roca [12, 13] used the least-squares approximation of an affine mapping for the projection of the *source* surface mesh onto the *target* surface. The extra term is added to the general least-squares approximation to avoid the skewn and flattening effects during the mesh sweeping. However, this approach still suffers from poor mesh quality for *source*/*target* surfaces which are concave or non-simply-connected. The *BoundaryError* method was introduced to place the interior nodes between the *source* surfaces and *target* surfaces using the linear affine algorithm and a subsequent residual error correction [9, 18]. M. L. Staten et al. [11] developed a new algorithm *BMSweep* to place the interior nodes while volume sweeping. The background mesh generated by performing the *Constrained Delaunay Triangulation* for the boundary nodes is used. However, the same mesh connectivity for all the layers will produce the inverted elements for the background mesh if there is a twisted feature in the model.

Therefore, based on the concept of *morphing*, both *source* surface and *target* surface can be mapped to a common domain, which is usually a convex polygon such as the unit disk. In this paper, *harmonic mapping* is used to map the *source* surface and *target* surface onto a unit disk. *Harmonic mapping* has many merits for surface mapping [18, 14]: (1) *Harmonic mapping* is computed through the global optimization and takes into account the surface topology. Therefore, local minimal, folding and clustering can be avoided; (2) insensitive to the resolution of face surface and noises on the surface; (3) it does not require the surface to be smooth. Even there is a sharp feature on the surface, it can be accurately computed. ; (4) Since we map the surface onto a unit disk, *harmonic mapping* exists and is diffeomorphism; (5) it is determined by the metric, not the embedding. This indicates that *harmonic mapping* is invariant for the same surface with different orientations. As long as there is not too much stretching between two surfaces with different mathematical expressions, they will induce similar harmonic maps.

3 Harmonic Mapping

In this paper, the application of *harmonic mapping* on projecting *source* mesh onto the *target* surface is proposed. In one-to-one sweeping, the *source* surface and *target* surface are topologically equivalent. These surfaces are represented as a triangular mesh in graphics. Therefore we call them the *source* surface M_1 and the *target* surface M_2, respectively. However, they may have different mesh nodes and connectivity. It takes two steps to transform from M_1 to M_2. The first step is to establish the correspondence map between M_1 and M_2, and then the interior nodes are interpolated accordingly.

The *Harmonic Mapping* develops M_1 and M_2 to the 2D unit disks [15, 16, 17], which we call H_1 and H_2, respectively. H_1 and H_2 have the same mesh connectivity as M_1 and M_2, respectively. H_1 and H_2 are created by mapping M_1 and M_2 onto the unit disk via *harmonic mapping*, respectively. Then the correspondence between H_1 and H_2 is generated. In order to make the correspondence, a new common 2D unit disk H_c needs to be created by adjusting the nodes' location on boundaries and combining both H_1 and H_2 (H_c has both M_1's connectivity and M_2's connectivity). For the sake of simplicity, we keep 2D unit disk H_2 fixed. Without creating a new H_c and mapping H_1 and H_2 to H_c, the boundary nodes of H_1 are adjusted in order to make the corresponding nodes between H_1 and H_2 overlap. After boundary correspondence between H_1 and H_2 is made, 2D unit disk H_1 is mapped onto H_2 directly. Then correspondence between M_1 and M_2 is established. After that, the all-quad mesh on the *source* surface can be mapped onto the *target* surface. In order to guarantee that *harmonic mapping* is one-to-one and well defined for geometries with large aspect ratios, the triangular mesh should be smoothed before *harmonic mapping*. The details are shown in **Fig** 1.

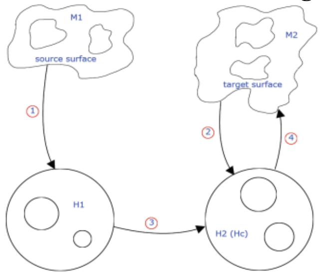

Fig. 1. Road map for mapping all-quad mesh from the *source* surface to the *target* surface.

4 Establish the surface correspondence

In the Sect 3, two embeddings H_1 and H_2 are created from the *source* surface M_1 and *target* surface M_2, respectively. In this section, two embeddings H_1 and H_2 are merged into H_2, which has combined mesh connectivity from both *source* surface M_1 and *target* surface M_2. After correspondence between H_1 and H_2 is made, any point from *source* surface M_1 corresponds with a point on the *target* surface M_2. All-quad mesh generation on the *target* surface consists of four steps.

In the first step, rotate H_1(or H_2) around the center of 2D unit disk so that the known corresponding vertices on the outmost boundary of *source* surface M_1 and *target* surface M_2 overlap (corresponding vertices can be obtained from *linking* sides between the *source* surface and *target* surface).

In the second step, if the surface is non-simply connected, adjust the known corresponding vertices (in H_1 or H_2) on interior boundaries of *source* surface M_1

and *target* surface M_2 to ensure that they overlap. Based on the same procedure as *Harmonic Mapping*, redistribute the interior vertices in H_1. Then smooth H_1 and H_2 with vertices on the boundaries fixed.

In the third step, calculate the corresponding 3D positions on *source* surface M_1 of every mesh node at H_1. Because there is an all-quad mesh on the *source* surface, a triangular face is searched at H_1 on the *source* surface where each mesh node n_m^1 is located in Fig. 2(a). When n_m^1 is located in a face (v_i^1, v_j^1, v_k^1) of H_1, the barycentric coordinates can be computed as follows.

$$\begin{aligned} n_m^1 &= i_1^1 v_i^1 + i_2^1 v_j^1 + i_3^1 v_k^1 \\ i_1^1 + i_2^1 + i_3^1 &= 1 \end{aligned} \tag{1}$$

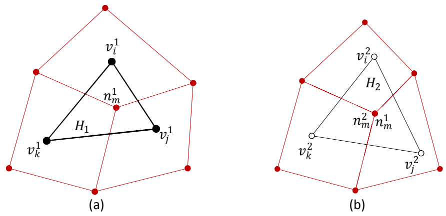

Fig. 2. Mapping vertex n_m^1 in H_1 to H_2

Finally, calculate corresponding 3D positions on the *target* surface M_2 of each mesh node n_m^2 at H_2. Search a face at H_2 where a node n_m^2 in H_1 is included in order to compute the 3D location of node n_m^2 on *target* surface M_2 in **Fig**. 2(b). When n_m^1 is located in a face (v_i^2, v_j^2, v_k^2) at H_2, the barycentric coordinates (i_1^2, i_2^2, i_3^2) can be computed as follows.

$$\begin{aligned} n_m^2 &= i_1^2 v_i^2 + i_2^2 v_j^2 + i_3^2 v_k^2 \\ i_1^2 + i_2^2 + i_3^2 &= 1 \end{aligned} \tag{2}$$

An example with concave and multi-connected features is shown in **Fig** 3.

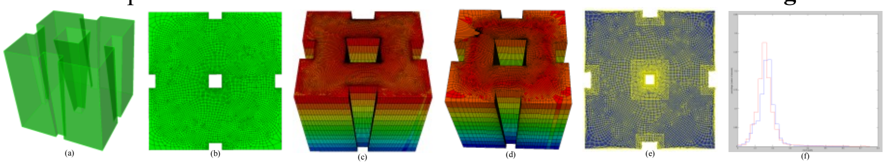

Fig. 3. Sweep volume with concavities and varying square holes: (a) the 3D model; (b) the *source* mesh (c) an all-hex mesh by our method; (d) an all-hex mesh from *Cubit* 12.2 which contains a lot of inverted elements; (e) an all-hex mesh from linear method which contains many inverted elements; (f) the mesh quality histogram for the result from our method(blue) and *Cubit* 12.2(red)

5 Conclusion

In this paper, a new algorithm to project all-quad mesh on *source* surface to *target* surface based on *harmonic mapping* has been developed. The projection between two topologically equivalent surfaces is determined by making the correspondence between them based on *harmonic mapping* by three steps. Firstly, we generate 2D unit disk for two surfaces based on *harmonic mapping*. Secondly, we make the correspondence between two unit disks. Finally, all-quad mesh on the *source* surface is mapped back to the *target* surface. The result shows that the all-quad mesh on the *source* surface can be mapped onto the *target* surface without any inverted element.

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