Application of Harmonic Mapping on One-To-One Sweeping

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Abstract. The sweeping algorithm can generate all-hexahedral mesh by sweeping an all-quad mesh on the source surface to the target surface. For one-to-one sweeping, the most difficult thing is to generate an all-quad mesh on the target surface which has the same mesh connectivity as that of source surface. The traditional method is to use the affine transformations, like translation, rotation, scaling, or combinations of them. This method works very well on the convex cases, while it fails for the concave and non-simply connected cases. In this paper, harmonic mapping is used to map an all-quad mesh onto the target surface. The result shows that it can generate an all-quad mesh on the target surface without any inverted element and avoid expensive smoothing algorithm like untangling.

1 Introduction

In many applications such as computational fluid dynamics (CFD) [1], hex meshes are preferred over the tetrahedral meshes. According to [2], there are two classes of methods for generating all-hex meshes, namely, indirect methods, which convert from tetrahedral to hexahedral meshes [8], and direct methods. The latter may be further classified as Grid-Based [3], medial surface [4, 5], plastering [6] and Whisker Weaving [7]. Because it is difficult to combine or divide the tetrahedron in such a way to guarantee the formation of all-hex mesh, the indirect methods are neither reasonable nor tractable for the mesh generation [2]. For the Grid-Based method, mesh quality at the boundary of volume is very poor and interior hex elements are not aligned with boundary hex elements, which don’t qualify the ideal requirements for CFD. The medial surface generates the hex meshes by decomposing the volume, which is an extension of medial axis method. The decomposed volumes are usually meshed with midside subdivision. However, this only works for the geometry with 3-valent corner vertices, and lacks reliability for the general geometry. Plastering is a 3D extension of the paving algorithm. Whisker Weaving method builds the dual of the hexahedral mesh and embeds the mesh in 3D. Neither plastering nor Whisker Weaving has been shown to be robust for the general 3D models.

While all-hexahedral mesh generation on the general 3D geometry remains an elusive goal, algorithms to mesh two-and-one-half dimensional geometries, generally referred to as “sweeping” or “projection” methods, continue to be important [10, 13]. The traditional one-to-one sweeping procedure consists of four steps: (1) generate an all-quad mesh on the source surface; (2) project an all-quad mesh
from source surface to the target surface; (3) generate the structured all-quad meshes on the linking surfaces; (4) generate an all-hex mesh, including interior nodes and elements. The source and target surfaces may have different shapes, areas or curvatures, but they must be topologically equivalent. Of the above four steps, the most difficult one is to map of source surface mesh to the target surface. This is especially true for models with concave or non-simply connected features.

2 Previous Work.

P. Knupp [10] devised two new algorithms to locate the interior nodes: linear transformations between the bounding node loops and smoothing. For the concave and non-simply connected source or target surfaces, this approach does not work. X. Roca [12, 13] used the least-squares approximation of an affine mapping for the projection of the source surface mesh onto the target surface. The extra term is added to the general least-squares approximation to avoid the skew and flattening effects during the mesh sweeping. However, this approach still suffers from poor mesh quality for source/target surfaces which are concave or non-simply-connected. The BoundaryError method was introduced to place the interior nodes between the source surfaces and target surfaces using the linear affine algorithm and a subsequent residual error correction [9, 18]. M. L. Staten et al. [11] developed a new algorithm BMSweep to place the interior nodes while volume sweeping. The background mesh generated by performing the Constrained Delaunay Triangulation for the boundary nodes is used. However, the same mesh connectivity for all the layers will produce the inverted elements for the background mesh if there is a twisted feature in the model.

Therefore, based on the concept of morphing, both source surface and target surface can be mapped to a common domain, which is usually a convex polygon such as the unit disk. In this paper, harmonic mapping is used to map the source surface and target surface onto a unit disk. Harmonic mapping has many merits for surface mapping [18, 14]: (1) Harmonic mapping is computed through the global optimization and takes into account the surface topology. Therefore, local minimal, folding and clustering can be avoided; (2) insensitive to the resolution of face surface and noises on the surface; (3) it does not require the surface to be smooth. Even there is a sharp feature on the surface, it can be accurately computed. (4) Since we map the surface onto a unit disk, harmonic mapping exists and is diffeomorphism; (5) it is determined by the metric, not the embedding. This indicates that harmonic mapping is invariant for the same surface with different orientations. As long as there is not too much stretching between two surfaces with different mathematical expressions, they will induce similar harmonic maps.

3 Harmonic Mapping

In this paper, the application of harmonic mapping on projecting source mesh onto the target surface is proposed. In one-to-one sweeping, the source surface and target surface are topologically equivalent. These surfaces are represented as a
triangular mesh in graphics. Therefore we call them the source surface $M_1$ and the target surface $M_2$, respectively. However, they may have different mesh nodes and connectivity. It takes two steps to transform from $M_1$ to $M_2$. The first step is to establish the correspondence map between $M_1$ and $M_2$, and then the interior nodes are interpolated accordingly.

The Harmonic Mapping develops $M_1$ and $M_2$ to the 2D unit disks [15, 16, 17], which we call $H_1$ and $H_2$, respectively. $H_1$ and $H_2$ have the same mesh connectivity as $M_1$ and $M_2$, respectively. $H_1$ and $H_2$ are created by mapping $M_1$ and $M_2$ onto the unit disk via harmonic mapping, respectively. Then the correspondence between $H_1$ and $H_2$ is generated. In order to make the correspondence, a new common 2D unit disk $H_3$ needs to be created by adjusting the nodes’ location on boundaries and combining both $H_1$ and $H_2$ ($H_3$ has both $M_1$’s connectivity and $M_2$’s connectivity). For the sake of simplicity, we keep 2D unit disk $H_2$ fixed. Without creating a new $H_3$ and mapping $H_1$, $H_2$ to $H_3$, the boundary nodes of $H_1$ are adjusted in order to make the corresponding nodes between $H_1$ and $H_2$ overlap. After boundary correspondence between $H_1$ and $H_2$ is made, 2D unit disk $H_3$ is mapped onto $H_2$ directly. Then correspondence between $M_1$ and $M_2$ is established. After that, the all-quad mesh on the source surface can be mapped onto the target surface. In order to guarantee that harmonic mapping is one-to-one and well defined for geometries with large aspect ratios, the triangular mesh should be smoothed before harmonic mapping. The details are shown in Fig. 1.

![Fig. 1. Road map for mapping all-quad mesh from the source surface to the target surface.](image)

### 4 Establish the surface correspondence

In the Sect 3, two embeddings $H_1$ and $H_2$ are created from the source surface $M_1$ and target surface $M_2$, respectively. In this section, two embeddings $H_1$ and $H_2$ are merged into $H_3$, which has combined mesh connectivity from both source surface $M_1$ and target surface $M_2$. After correspondence between $H_1$ and $H_2$ is made, any point from source surface $M_1$ corresponds with a point on the target surface $M_2$. All-quad mesh generation on the target surface consists of four steps.

In the first step, rotate $H_1$ (or $H_2$) around the center of 2D unit disk so that the known corresponding vertices on the outmost boundary of source surface $M_1$ and target surface $M_2$ overlap (corresponding vertices can be obtained from linking sides between the source surface and target surface).

In the second step, if the surface is non-simply connected, adjust the known corresponding vertices (in $H_1$ or $H_2$) on interior boundaries of source surface $M_1$
and target surface $M_2$ to ensure that they overlap. Based on the same procedure as Harmonic Mapping, redistribute the interior vertices in $H_1$. Then smooth $H_1$ and $H_2$ with vertices on the boundaries fixed.

In the third step, calculate the corresponding 3D positions on source surface $M_1$ of every mesh node at $H_1$. Because there is an all-quad mesh on the source surface, a triangular face is searched at $H_1$ on the source surface where each mesh node $n_m^w$ is located in Fig. 2(a). When $n_m^w$ is located in a face $(v_i^w, v_j^w, v_k^w)$ of $H_1$, the barycentric coordinates can be computed as follows.\begin{equation}
\begin{align*}
n_m^w &= i_1^w v_i^w + i_2^w v_j^w + i_3^w v_k^w \\
i_1^w + i_2^w + i_3^w &= 1
\end{align*}
\end{equation}

Finally, calculate corresponding 3D positions on the target surface $M_2$ of each mesh node $n_m^w$ at $H_2$. Search a face at $H_2$ where a node $n_m^w$ in $H_1$ is included in order to compute the 3D location of node $n_m^w$ on target surface $M_2$ in Fig. 2(b). When $n_m^w$ is located in a face $(v_i^w, v_j^w, v_k^w)$ at $H_2$, the barycentric coordinates $(i_1^w, i_2^w, i_3^w)$ can be computed as follows.
\begin{equation}
\begin{align*}
n_m^w &= i_1^w v_i^w + i_2^w v_j^w + i_3^w v_k^w \\
i_1^w + i_2^w + i_3^w &= 1
\end{align*}
\end{equation}

An example with concave and multi-connected features is shown in Fig. 3.

5 Conclusion

In this paper, a new algorithm to project all-quad mesh on source surface to target surface based on harmonic mapping has been developed. The projection between
two topologically equivalent surfaces is determined by making the correspondence between them based on harmonic mapping by three steps. Firstly, we generate 2D unit disk for two surfaces based on harmonic mapping. Secondly, we make the correspondence between two unit disks. Finally, all-quad mesh on the source surface is mapped back to the target surface. The result shows that the all-quad mesh on the source surface can be mapped onto the target surface without any inverted element.

References