Defining quality measures for mesh optimization on parameterized CAD surfaces

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Summary. We present a technique to extend any distortion (quality) measure for planar meshes to meshes on parameterized surfaces. The resulting distortion (quality) measure is expressed in terms of the parametric coordinates of the nodes. This extended distortion (quality) measure can be used to check the quality and validity of a surface mesh. We also apply it to simultaneously smooth and untangle surface meshes by minimizing the extended distortion measure. The minimization is performed in terms of the parametric coordinates of the nodes and therefore, the nodes always lie on the surface. In addition, we apply the extension technique to define quality measures for high-order meshes on parameterized surfaces. Finally, we include some examples to show the application of the proposed technique. Specifically, we extend several Jacobian based measures, and we smooth and untangle linear and high-order meshes on CAD surfaces.

1 Introduction

In the last decades, computational methods have shown to be powerful tools to solve partial differential equations in applied sciences and engineering. To apply these methods it is required to generate a mesh of the domain. The mesh has to be composed by non-inverted (valid) and well-shaped elements (quality) [1, 2, 3]. If one element is inverted, the variational formulation for that mesh is not valid. Moreover, just a few low-quality elements can compromise the accuracy of the solution in the whole domain. Therefore, quality measures have to be used to determine the validity of a mesh.

Given a mesh with low-quality elements, a standard procedure to improve the quality of the mesh is to relocate the node positions (smoothing) [4, 5, 6]. However, special attention has to be paid to inverted elements in the mesh. First, if the initial mesh contains inverted elements, few smoothing methods can achieve a valid configuration (untangle). Second, some smoothing methods can obtain inverted configurations, specially when the boundary is non-convex. Therefore, several methods have been developed to untangle meshes [7, 8, 9]. Note that combining an untangling method with a smoothing technique, we can obtain the desired valid and high-quality mesh [10].

In order to smooth a mesh, we require distinguishing between boundary and inner nodes. Whereas interior nodes can move freely inside the container volumes, boundary nodes can only move on the surface where they lie. Moreover, if a boundary mesh face is inverted, the corresponding mesh element is inverted. Therefore, it is of the major importance to ensure a high-quality surface mesh in order to be able to define a high-quality 3D mesh. To be able to optimize a surface mesh, we require to involve the geometry representation in the optimization procedure. Several geometry representations can be used: triangular mesh, implicit entity, or CAD entities are the most common techniques. For industrial applications the CAD surface description is preferred, since CAD models are generated in the design process.

The main contribution of this work is to present a technique to extend any distortion (quality) for planar elements to elements with the nodes on parameterized surfaces (CAD). Then, the resulting measures are expressed in terms of the parametric coordinates of the surface. In addition, this technique allows the extension of measures for linear [1, 2, 3] and high-order [11] planar elements to elements on parameterized surfaces. We apply this technique to smooth and untangle linear and high-order triangular meshes with the nodes on a parameterized surface. The resulting meshes are composed by valid and high-quality elements with the nodes on the surface. In addition, we can ensure the optimized nodes lie on the original CAD surface and not on an approximation of it. To this end, we use the surface parameterization to map the optimal parametric coordinates to points on the CAD surface.

The rest of the paper is organized as follows. First, in Section 2 we review the related work on surface mesh optimization. In Section 3, we extend any distortion and quality measure for linear elements to elements with the nodes on parameterized surfaces. Then, in Section 4, we detail the optimization procedure in terms of the parametric coordinates. We develop a non-linear least-squares problem in order to enforce the ideal configuration for the elements of the surface mesh, Section 4.1. The implementation of the method is detailed in Section 4.2. For completeness, in Section 4.3 we detail how to incorporate several Jacobian-based distortion measures in the optimization algorithm. In Section 5, we use the presented technique to extend measures for planar high-order elements to elements on parameterized surfaces. Finally, we present several examples to show the applications of the proposed method, Section 6.

2 Related work

Quality measures are defined to quantify the geometrical validity of the elements of a mesh, see [1] for a comparative analysis. In this work, we focus on the framework of algebraic quality measures introduced by Knupp [2, 3]. The main idea of this framework is to define the quality (distortion) of an element as a measure of the deviation of the physical element with respect to an ideal element. The resulting quality measures are expressed in terms of the physical coordinates of the element nodes. In this work, we express the quality measures in terms of the parametric coordinates of the vertices.

In order to improve the quality of a valid mesh, an optimization approach based on Jacobian-based measures is proposed in [6]. Optimization based approaches can also be used to untangle inverted elements. In particular, Freitag proposed a two-step procedure [7]: an untangling step to define a valid mesh, followed by an optimization approach. Similar approaches divided in two steps have been developed [8, 9]. Later, Escobar *et al.* proposed a simultaneous smoothing and untangling technique [10]. This technique has been also extended to non-planar triangular meshes [12, 13]. In our work, we use the modification of the element distortion proposed in [10].

Two main approaches have been proposed to relocate nodes on surface meshes. On the one hand, several methods compute an ideal location of the optimized node, that can be off the surface, and then relocate the nodes on the surface [12, 13, 14, 15, 16]. On the other hand, there also exist several methods that obtain an ideal location of the nodes directly on the surface [17, 18, 19]. These methods, express the optimization procedure in terms of the parametric coordinates of an approximated representation of the original surface. We also compute the optimal location directly on the surface. However, we propose to quantify the distortion (quality) of the element in terms of the coordinates on the parametric space of the CAD surface. An optimization approach based on the proposed distortion ensures that the nodes always lie on the surface, since the whole process is developed in the parametric space of the original surface.

3 Distortion and quality for elements on parameterized surfaces

In this section, we first develop an analytical formulation to extend any quality measure for planar triangles to triangular meshes on a parameterized surface. As a result, we obtain a quality measure expressed in the two coordinates of the parametric space of the surface.

Note that triangle elements on a surface are two-dimensional planar entities immersed in $\mathbb{R}^3$. Hence, a possible approach to qualify surface elements is to use planar quality measures in the plane where the surface element lies. However, this approach does not allow a straight forward surface optimization procedure. Parameterized surfaces enable the development of an analytical and straightforward function to quantify the quality of a surface triangular element. Herein, we propose a composition of the parameterization of the surface together with a mapping of the surface element to a similar one in a 2D space, where planar quality measures are defined. This way, the proposed quality is expressed in terms of the parametric coordinates of the surface. Such expression allows a natural smoothing technique that avoids any additional constraint to keep the nodes on the surface, since the procedure is developed on its parametric space.

3.1 Preliminaries

Let η be a distortion measure for planar elements, with image $[1,\infty)$, taking value 1 for an ideal configuration of the element, and value ∞ when it is degenerated or tangled. Let q be the corresponding quality measure, defined as

$$q = \frac{1}{\eta}. \tag{1}$$

The image of the quality measure q is $[0,1]$, taking value 1 for ideal configurations and 0 for degenerated or tangled ones. These measures for planar elements presented can be expressed as the mappings

$$\eta : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [1,\infty) \subset \mathbb{R}, \tag{2}$$

$$q : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow [0,1] \subset \mathbb{R}. \tag{3}$$

Given a distortion and its associated quality measure for triangles in the plane, our goal is to extend these measure to triangles with the vertices on a parameterized surface, Σ. Assume that the surface Σ is parameterized by a continuously differentiable and invertible mapping

$$\begin{aligned} \varphi : \quad \mathcal{U} \subset \mathbb{R}^2 &\longrightarrow \quad \Sigma \subset \mathbb{R}^3 \\ \mathbf{u} = (u,v) &\longmapsto \mathbf{x} = \varphi(\mathbf{u}). \end{aligned} \tag{4}$$

It is important to point out that in our applications we consider parameterized CAD surfaces. Thus, the evaluation of the surface parameterization and its derivatives requires a geometry engine. Specifically, in our implementation we use OpenCASCADE library [20].

3.2 Measures for triangles on parametric coordinates

To evaluate the quality of a triangle t_Σ with vertices on a surface Σ, we first express the vertices as the image by the parameterization φ of the corresponding parametric coordinates in $\mathcal{U}$. Since t_Σ is planar, but it is immersed in $\mathbb{R}^3$, we define the quality of the physical triangle as the quality of a geometrically equivalent triangle t on $\mathbb{R}^2$. Once in $\mathbb{R}^2$, the proposed formulation allows to extend any existent distortion and quality measure for planar elements. In Section 4.3, we detail the distortion measures considered in this work.

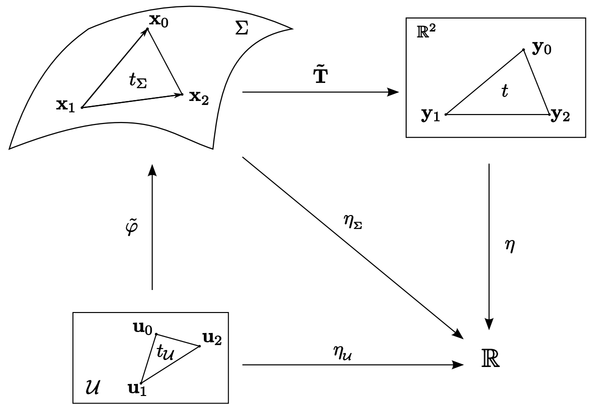

Fig. 1. Diagram of mappings involved in the definition of the quality measure.

In order to define a quality measure in terms of the parametric coordinates of the three vertices of the triangle, we define the mapping

$$\begin{aligned} \tilde{\varphi} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} &\longrightarrow \qquad\qquad \mathit{\Sigma} \times \mathit{\Sigma} \times \mathit{\Sigma} \\ (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2) &\longmapsto (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) = (\varphi(\mathbf{u}_0), \varphi(\mathbf{u}_1), \varphi(\mathbf{u}_2)). \end{aligned} \tag{5}$$

This mapping transforms a triangle $t_\mathcal{U} = (\mathbf{u}_0, \mathbf{u}_1, \mathbf{u}_2)$ in the parametric space $\mathcal{U}$, to a triangle $t_\mathit{\Sigma} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ with the nodes on the surface $\mathit{\Sigma}$ determined by φ, see Figure 1. Since $t_\mathit{\Sigma}$ defines a plane in $\mathbb{R}^3$, we can map $t_\mathit{\Sigma}$ to a geometrically equivalent triangle in $\mathbb{R}^2$. That is, we can define a mapping $\tilde{\mathbf{T}}$ from $\mathit{\Sigma} \times \mathit{\Sigma} \times \mathit{\Sigma}$ to $\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$. To define $\tilde{\mathbf{T}}$, we consider an auxiliary linear mapping $\mathbf{T}$ from $\mathbb{R}^3$ to the plane. The domain of this mapping is expressed in the canonical basis of $\mathbb{R}^3$, and the image is expressed in terms of a new 2D orthogonal basis determined by a combination of two edges of the triangle. Let

$$\begin{aligned} \mathbf{e}_1 &:= \mathbf{x}_2 - \mathbf{x}_1, \\ \mathbf{e}_2 &:= \mathbf{x}_0 - \mathbf{x}_1, \end{aligned} \tag{6}$$

be the vectors determined by two edges of the triangle. Then, we define

$$\begin{aligned} \tilde{\mathbf{e}}_1 &:= \frac{\mathbf{e}_1}{\|\mathbf{e}_1\|}, \\ \tilde{\mathbf{e}}_2 &:= \gamma\, \tilde{\mathbf{e}}_{2,0}, \qquad \text{with} \quad \tilde{\mathbf{e}}_{2,0} := \frac{\mathbf{e}_2 - (\mathbf{e}_2^T \cdot \tilde{\mathbf{e}}_1)\, \tilde{\mathbf{e}}_1}{\|\mathbf{e}_2 - (\mathbf{e}_2^T \cdot \tilde{\mathbf{e}}_1)\, \tilde{\mathbf{e}}_1\|}, \end{aligned}$$

as the two orthonormal vectors of the new basis, where γ is defined to ensure a well oriented orthonormal basis. Specifically, we define γ as:

$$\gamma := \frac{(\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_{2,0}) \cdot \mathbf{n}}{|(\tilde{\mathbf{e}}_1 \times \tilde{\mathbf{e}}_{2,0}) \cdot \mathbf{n}|} = \frac{\det(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_{2,0}, \mathbf{n})}{|\det(\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_{2,0}, \mathbf{n})|},$$

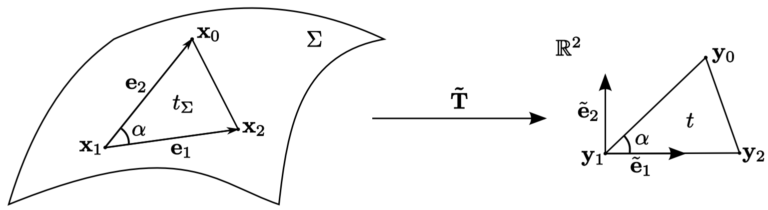

Fig. 2. Vector edges $\mathbf{e}_1$ and $\mathbf{e}_2$ for a triangle $t_\Sigma = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)$ on a surface Σ, and diagram of function $\tilde{\mathbf{T}}$.

where $\mathbf{n} \equiv \mathbf{n}(\mathbf{x}_1) = \frac{\partial \varphi}{\partial u}(u_1, v_1) \times \frac{\partial \varphi}{\partial v}(u_1, v_1)$ is the normal to the surface at $\mathbf{x}_1 = \varphi(u_1, v_1)$. Note that $\gamma = \pm 1$, being 1 for counter-clockwise oriented triangles, and -1 for clockwise oriented ones.

Now, we can define $\mathbf{T}$ as

$$\begin{aligned} \mathbf{T} : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ \mathbf{x} &\longmapsto \mathbf{M} \cdot (\mathbf{x} - \mathbf{x}_1), \end{aligned} \tag{7}$$

where $\mathbf{M} = (\tilde{\mathbf{e}}_1 \;\; \tilde{\mathbf{e}}_2)^T$ is a 2×3 matrix. In addition, we define $\tilde{\mathbf{T}}$ as:

$$\begin{aligned} \tilde{\mathbf{T}} : \quad \Sigma \times \Sigma \times \Sigma \quad &\longrightarrow \quad \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \\ t_\Sigma = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) &\longmapsto t = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = (\mathbf{T}(\mathbf{x}_0), \mathbf{T}(\mathbf{x}_1), \mathbf{T}(\mathbf{x}_2)), \end{aligned} \tag{8}$$

see Figure 2. Hence, we can express the distortion measure for a triangle t_Σ on the surface as:

$$\begin{aligned} \eta_\Sigma : \Sigma \times \Sigma \times \Sigma &\xrightarrow{\tilde{\mathbf{T}}} \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{\eta} \qquad \mathbb{R} \\ (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) &\longmapsto \tilde{\mathbf{T}}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2) \longmapsto \eta(\tilde{\mathbf{T}}(\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2)). \end{aligned}$$

That is, as the composition

$$\eta_\Sigma = \eta \circ \tilde{\mathbf{T}} : \Sigma \times \Sigma \times \Sigma \longrightarrow [1, \infty). \tag{9}$$

Note that η_Σ is a distortion measure on Σ, since it is the composition of a planar distortion measure η, and a change of variable of the plane where t_Σ lies. Moreover, the reciprocal of η_Σ,

$$q_\Sigma := \frac{1}{\eta_\Sigma} : \Sigma \times \Sigma \times \Sigma \longrightarrow [0, 1],$$

is also a quality measure, in the sense of [2]. It is important to point out that this quality measure holds the same properties of the corresponding original planar quality measure q.

Finally, we use the expression of the distortion η_Σ, Equation (9), to define the distortion and quality measures in terms of the parametric coordinates of the triangle.

Definition 1. *The distortion measure for triangles on parametric coordinates is:*

$$\eta_{\mathcal{U}} := \eta_{\Sigma} \circ \tilde{\varphi} = \eta \circ \tilde{\mathbf{T}} \circ \tilde{\varphi} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow [1, \infty). \tag{10}$$

Definition 2. *The quality measure for triangles on parametric coordinates is:*

$$q_{\mathcal{U}} := \frac{1}{\eta_{\mathcal{U}}} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \longrightarrow [0, 1]. \tag{11}$$

Next, we present a simplified expression for the distortion measure for triangles on parametric coordinates, see Equation (10), when there is one free node in the triangle and the rest of nodes are fixed.

Remark 1. Let $\mathbf{u}$ be the free node of a triangle and $\mathbf{u}_1$ and $\mathbf{u}_2$ the two fixed nodes. It can be proved that for fixed values of $\mathbf{u}_1$ and $\mathbf{u}_2$, the restriction of $\eta_{\mathcal{U}}$ to a free node $\mathbf{u}$,

$$\begin{array}{rcccc} \eta_{\mathcal{U}}(\cdot; \mathbf{u}_1, \mathbf{u}_2) : \mathcal{U} & \xrightarrow{\tilde{\varphi}(\cdot; \mathbf{u}_1, \mathbf{u}_2)} & \Sigma \times \Sigma \times \Sigma \xrightarrow{\tilde{\mathbf{T}}} & \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 & \xrightarrow{\eta} \mathbb{R} \\ \mathbf{u} & \longmapsto & \tilde{\varphi}(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2) \longmapsto & \tilde{\mathbf{T}}(\tilde{\varphi}(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2)) & \longmapsto \eta(\tilde{\mathbf{T}}(\tilde{\varphi}(\mathbf{u}, \mathbf{u}_1, \mathbf{u}_2))), \end{array}$$

corresponds to the expression

$$\begin{array}{rcccc} \hat{\eta}_{\mathcal{U}} : \mathcal{U} \subset \mathbb{R}^2 & \xrightarrow{\varphi} \Sigma \subset \mathbb{R}^3 & \xrightarrow{\mathbf{T}} & \mathbb{R}^2 & \xrightarrow{\hat{\eta}} \mathbb{R} \\ \mathbf{u} & \longmapsto \varphi(\mathbf{u}) & \longmapsto & \mathbf{T}(\varphi(\mathbf{u})) & \longmapsto \hat{\eta}(\mathbf{T}(\varphi(\mathbf{u}))), \end{array}$$

where

$$\hat{\eta}(\mathbf{y}) := \eta(\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2), \tag{12}$$

being straightforward to pre-coumpute the values $\mathbf{y}_1 := (0\ 0)^T$, and $\mathbf{y}_2 := (\|\mathbf{e}_1\|\ 0)^T$. That is, it corresponds to the composition

$$\hat{\eta}_{\mathcal{U}} = \hat{\eta} \circ \mathbf{T} \circ \varphi : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow [1, \infty), \tag{13}$$

referred as the *restricted distortion measure for triangles on parametric coordinates.*

4 Application: optimization of surface mesh quality

In this section, we present an algorithm to optimize the distortion (quality) measure of triangular and quadrilateral meshes on parameterized surfaces. First, we formulate the optimization problem. Second, we present the proposed implementation. Finally, we specify some implementation details.

4.1 Formulation: imposing an ideal mesh distortion in the least-squares sense

The main goal of a simultaneous smoothing and untangling method is to obtain high-quality meshes composed by valid (non-inverted) elements. Note that the best possible result, can be characterized in terms of the distortion measure. That is, given a distortion measure $\eta_{\mathcal{U}}$ and a mesh $\mathcal{M}$ on a parameterized surface composed by n_N nodes and n_E elements, the node location is ideal if

$$\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) = 1 \qquad j = 1, \ldots, n_E, \tag{14}$$

where $t^j_{\mathcal{U}} = (\mathbf{u}_{j_1}, \mathbf{u}_{j_2}, \mathbf{u}_{j_3})$ is the jth element expressed on parametric coordinates. However, for a fixed mesh topology the node location that leads to an ideal mesh distortion is not in general achievable. That is, the constraints in Equation (14) cannot be imposed strongly and therefore, we just enforce the ideal mesh distortion in the least-squares sense. It is important to point out that the best configuration is delimited by the initial mesh topology.

For a given mesh topology and a set of fixed nodes (nodes on the surface boundary), we formulate the least-squares problem in terms of the parametric coordinates of a set of free nodes (inner nodes on the surface). To this end, we reorder the parametric coordinates of the nodes, $\mathbf{u}_i$, in such a way that $i = 1, \ldots, n_F$ are the indices corresponding to the free nodes, and $i = n_F + 1, \ldots, n_N$ correspond to the fixed nodes. Thus, we can formulate the mesh optimization problem as

$$\min_{\mathbf{u}_1,\ldots,\mathbf{u}_{n_F}} f(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}), \tag{15}$$

where

$$f(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}) := \frac{1}{2}\sum_{j=1}^{n_E}(\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) - 1)^2$$

denotes the objective function.

Finally, the optimal configuration is found between the candidates for the minimization of (15). The candidates are the critical parametric coordinates $(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F})$ of f. They are characterized by ensuring, for $i = 1, \ldots, n_F$,

$$\frac{\partial f}{\partial \mathbf{u}_i}(\mathbf{u}_1, \ldots, \mathbf{u}_{n_F}; \mathbf{u}_{n_F+1}, \ldots, \mathbf{u}_{n_N}) = \sum_{j=1}^{n_E}(\eta_{\mathcal{U}}(t^j_{\mathcal{U}}) - 1)\ \frac{\partial \eta_{\mathcal{U}}}{\partial \mathbf{u}_i}(t^j_{\mathcal{U}}) = 0. \tag{16}$$

4.2 Implementation: deviation of the submesh distortion respect an ideal configuration

To solve the optimization problem in Equation (15), we have to find the optimum between the candidate configurations. These configurations are characterized by the global non-linear constraints in Equation (16). To solve these constraints, we choose a non-linear iterative method that: exploits the locality of the problem, avoids solving large linear systems, and is well suited for parallelization (by coloring the mesh nodes). Specifically, we use a non-linear iterative Gauss-Seidel method determined by the iteration

$$\mathbf{u}_i^{k+1} = \mathbf{u}_i^k - \alpha_i^k \; [\nabla^2_{ii} f(\mathbf{w}_i^k)]^{-1} \; \boldsymbol{\nabla}_i f(\mathbf{w}_i^k) \qquad i = 1, \ldots, n_F, \tag{17}$$

where α_i^k is the step length, and

$$\mathbf{w}_i^k = (\mathbf{u}_1^{k+1}, \ldots, \mathbf{u}_{i-1}^{k+1}, \mathbf{u}_i^k, \mathbf{u}_{i+1}^k, \ldots, \mathbf{u}_{n_F}^k; \mathbf{u}_{n_F+1}^0, \ldots, \mathbf{u}_{n_N}^0)$$

is the vector of updated node locations for the $i-1$ first nodes. Note that $\boldsymbol{\nabla}_i$ and ∇^2_{ii} denote the gradient and the Hessian with respect to the parametric coordinates $\mathbf{u}_i$ of node i.

In our implementation, we exploit the fact that each node only contributes to its neighbouring elements. To this end, we denote by $\mathcal{M}_{\mathbf{u}}$ the elements that contain a free node $\mathbf{u}$. The set of elements $\mathcal{M}_{\mathbf{u}}$ is referred as the *submesh* associated with node $\mathbf{u}$. Let $\mathbf{u}_i^k$ be the parametric coordinates of node i at step k, and let $\mathcal{M}_{\mathbf{u}_i^k}$ be the corresponding associated submesh composed by m_i elements. Let $\hat{\eta}^j_{\mathcal{U}}(\mathbf{u}_i)$ be the restricted distortion measure on the jth element of $\mathcal{M}_{\mathbf{u}_i^k}$, see Equation (13). We say that

$$\hat{f}(\mathbf{u}_i) = \sum_{j \sim i} \left(\hat{\eta}^j_{\mathcal{U}}(\mathbf{u}_i) - 1\right)^2 = \sum_{j=1}^{m_i} \left(\hat{\eta}^j_{\mathcal{U}}(\mathbf{u}_i) - 1\right)^2 \tag{18}$$

is a local merit function that measures the *deviation respect an ideal configuration of the submesh distortion associated with* $\mathbf{u}_i$, where $j \sim i$ denotes that the summation is performed only for the elements that contain the node i. According to this merit function we can implement the iteration $k+1$ for node i of the proposed non-linear method, Equation (17), as

$$\mathbf{u}_i^{k+1} = \mathbf{u}_i^k - \alpha_i^k \; [\nabla^2_{ii} \hat{f}(\mathbf{u}_i^k)]^{-1} \; \boldsymbol{\nabla}_i \hat{f}(\mathbf{u}_i^k) \qquad i = 1, \ldots, n_F. \tag{19}$$

4.3 Inclusion of several distortion measures

We apply the presented approach to define distortion and quality measures on parameterized surfaces to two Jacobian algebraic distortion measures for planar elements, presented in [2]. Specifically, we consider the shape and the Oddy measures. Moreover, we detail how to modify these measures to incorporate the untangling capability to the optimization method. To this end, we use the modification presented in [10]. This modification can be applied to distortion measures where the determinant of the Jacobian appears in the denominator.

In order to define a Jacobian-based measure for triangles, three types of elements are required: the reference, the ideal, and the physical. The reference element has an auxiliary use, since it is straight forward to define a linear

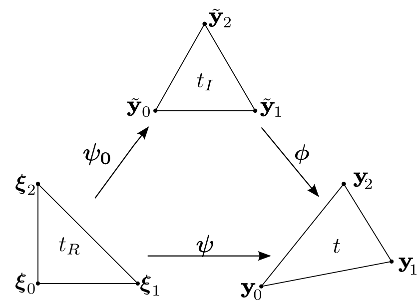

Fig. 3. Mappings between the reference, the ideal and the physical elements.

affine mapping between the reference and any other triangle. The ideal triangle represents the best configuration of the geometrical property to quantify. The physical is the element to be measured. Once the mappings between the reference and the ideal and the physical elements are obtained, a mapping between the ideal and the physical elements is determined by (see Figure 3)

$$\boldsymbol{\phi} : t_I \xrightarrow{\boldsymbol{\psi}_0^{-1}} t_R \xrightarrow{\boldsymbol{\psi}} t.$$

The Jacobian of this affine mapping contains information about the deviation of the physical element with respect to the ideal. Hence, the distortion measure of the physical element is defined in terms of $\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \mathbf{D}\boldsymbol{\phi}$. These distortion measures quantify the deviation of a geometrical property of the physical element respect the ideal element in a range scale $[1, \infty)$.

Shape distortion measure

$$\eta_{sh}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \frac{\|\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)\|^2}{2|\sigma(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2)|}, \tag{20}$$

where $\|\cdot\|$ is the Frobenius norm, and $\sigma(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \det(\mathbf{S}(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2))$. This distortion measure quantifies the deviation of the shape of the physical triangle with respect to the ideal shape. To incorporate the untangling capability to the optimization method, we replace σ in (20) by

$$\sigma_*(\sigma; \delta) = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right), \tag{21}$$

where δ is a numerical parameter that has to be determined [10].

Oddy measure

$$\eta(\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2) = \frac{3}{2}\sigma^{-2}\left(\|\mathbf{S}^T\mathbf{S}\|^2 - \frac{1}{3}\|\mathbf{S}\|^4\right), \tag{22}$$

where analogously to than for the shape distortion measure, we replace σ by σ_* to optimize tangled meshes.

5 Application: measures for high-order surface meshes

Finally, we show that the proposed technique can also be applied to define measures for high-order meshes on parameterized surfaces. Let η be a Jacobian distortion measure for linear elements (see Section 4.3), t a triangle with area $|t|$, and $\boldsymbol{\phi}$ the mapping between the ideal and the physical triangle. In [11] we propose the following definition of the distortion measure for a planar high-order triangular element:

Definition 3. *The* distortion measure for a high-order planar triangle *with nodes* $\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}$ *is*

$$\eta_{\boldsymbol{\phi}}(\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}) := \left(\frac{1}{|t|} \int_t \eta^2(\mathbf{D}\boldsymbol{\phi}(\boldsymbol{\phi}^{-1}(\mathbf{x}); \mathbf{x}_1, \ldots, \mathbf{x}_{n_p}))\ d\mathbf{x} \right)^{\frac{1}{2}}. \tag{23}$$

Similarly to the procedure developed for planar linear triangles, the previous measure for planar high-order triangles can be extended to parameterized surfaces. Let t_Σ be a triangle with the nodes on a surface parameterized by φ, $|t_\Sigma|$ the area of the triangle, and $\mathbf{T}$ the projection from the canonical basis on $\mathbb{R}^3$ to an orthonormal basis on the tangent plane for each point of the physical element. We consider the following definition:

Definition 4. *The* distortion measure for a high-order triangle on a parameterized surface Σ *with nodes* $\mathbf{u}_1, \ldots, \mathbf{u}_{n_p} \in \mathcal{U}$ *in the parametric space is*

$$\hat{\eta}_\varphi(\mathbf{u}_1, \ldots, \mathbf{u}_{n_p}) := \left(\frac{1}{|t_\Sigma|} \int_{t_\Sigma} \eta^2 \left(\mathbf{T} \left(\mathbf{D}\boldsymbol{\phi} \left(\boldsymbol{\phi}^{-1}(\mathbf{x}); \varphi(\mathbf{u}_1), \ldots, \varphi(\mathbf{u}_{n_p}) \right) \right) \right) d\mathbf{x} \right)^{\frac{1}{2}}.$$

This extended measure for high-order elements can be used to optimize high-order measures on parameterized surfaces. To this end, we extend to high-order meshes the optimization procedure presented in Section 4.1.

6 Numerical examples: optimization of surface mesh quality

In this section, we present several examples in order to show several properties of the proposed method: *1.* it is consistent with the original planar distortion measure, *2.* it can incorporate several planar distortion measures, *3.* it provides high-quality meshes even though the initial mesh contains a large number of tangled elements, *4.* it can be used to optimize surface meshes generated from industrial CAD models, and *5.* it can be used to optimize high-order meshes on CAD surfaces. For all the examples, we display a table summarizing the quality statistics of the meshes. Specifically, we provide: the minimum, the maximum, the mean and the standard deviation of the quality

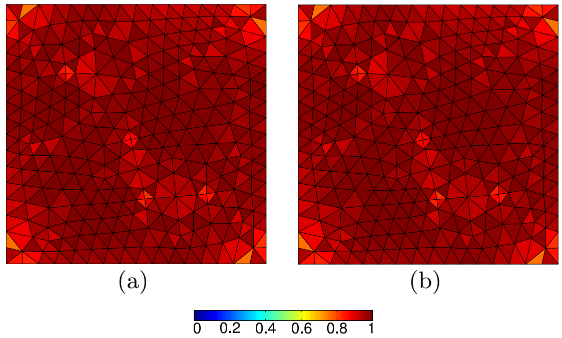

Fig. 4. Triangle mesh for a planar square. Mesh colored according to (a) planar shape quality measure, and (b) surface shape quality measure.

Table 1. Planar quality shape and surface quality shape statistics of the mesh on a planar square.

Measure	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Planar shape	4(a)	0.75	1.00	0.96	0.04	0
Surface shape	4(b)	0.75	1.00	0.96	0.04	0

of the elements, and the number of tangled elements. We highlight that in all cases, the smoothed mesh increases the minimum and mean values of the mesh quality and decreases its standard deviation. All algorithms have been implemented in C++ in the meshing environment EZ4U [21, 22, 23]. The goal of these examples is to show several applications of the proposed framework. In this sense, the current implementation is not totally optimized. Therefore, we do not include the CPU time required to optimize the presented examples.

6.1 Consistency

The goal of this example is to show that a planar distortion measure η and the corresponding extended surface distortion measure $\eta_{\mathcal{U}}$ are the same when the surface is the Euclidean plane. It is important to point out, that this is true by construction. That is, the parameterization of the plane is the identity, and the mapping $\mathbf{T}$ is just a rigid body motion. Since η is invariant under rigid body motion, we have that η is equal to $\eta_{\mathcal{U}}$. Specifically, in this example we consider η to be the shape distortion measure, see Figure 4. In addition, Table 1 shows the statistics corresponding to both measures. We observe that, as expected, both measures present the same results.

6.2 Several quality measures

The goal of this example is to show that any planar distortion measure can be extended to parameterized surfaces. Then, we use the technique presented in

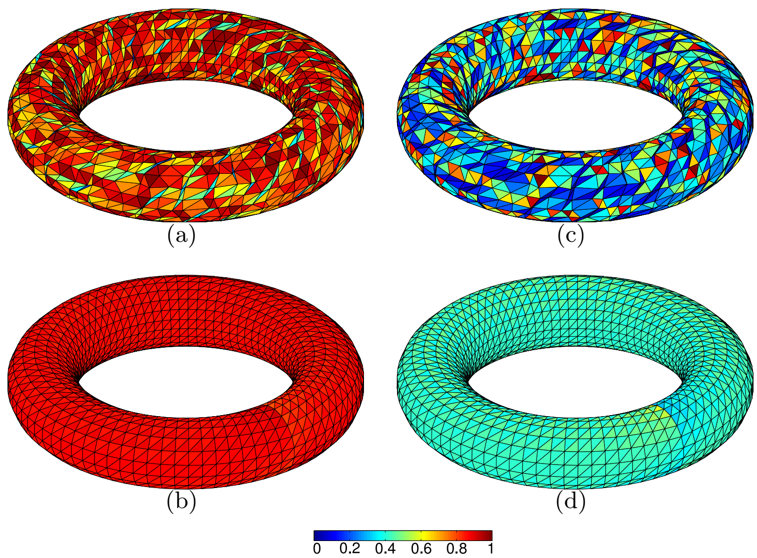

Fig. 5. Meshes for a torus. Meshes colored according to the shape quality measure: (a) initial mesh, and (b) smoothed and untangled mesh. Meshes colored according to the Oddy quality measure: (c) initial mesh, and (d) smoothed and untangled mesh.

Section 4 to optimize the quality of the mesh. However, note that this untangling technique can be only applied for Jacobian-based distortion measures with the determinant of the Jacobian on the denominator. Therefore, in this example we extend to parameterized surfaces only the shape and the Oddy measures. Then, we generate a triangular mesh on a torus composed by 1600 nodes and 3002 elements. In Figures 5(a) and 5(c) we show the initial mesh, coloring the elements with respect to the two different selected measures. Note that the mesh contains 73 inverted elements. Then, in Figures 5(b) and 5(d) we show the two resulting optimized meshes.

Table 2 summarizes the quality statistics of the meshes presented in Figure 5. Note that the proposed algorithm untangles an input mesh with inverted elements. In addition, for both cases, the proposed method improves the quality of the initial surface meshes. Note that the Oddy measure is more restrictive. That is, Oddy measure quantifies as low quality the rectangular triangles (the ideal triangle is the equilateral). Nevertheless, both measures properly detect the degenerated and the valid elements.

6.3 Surfaces composed of multiple patches

In this example, we apply the smoothing and untangling procedure presented in Section 4 using the shape distortion measure, Equation (20), to two CAD models composed by multiple patches: a component of a gear box and a crank

Table 2. Shape and Oddy quality statistics of the meshes on the torus.

Measure	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Shape	Tangled	5(a)	0.00	1.00	0.72	0.24	73
	Smoothed	5(a)	0.79	0.92	0.86	0.02	0
Oddy	Tangled	5(c)	0.00	1.00	0.34	0.26	73
	Smoothed	5(d)	0.31	0.59	0.42	0.03	0

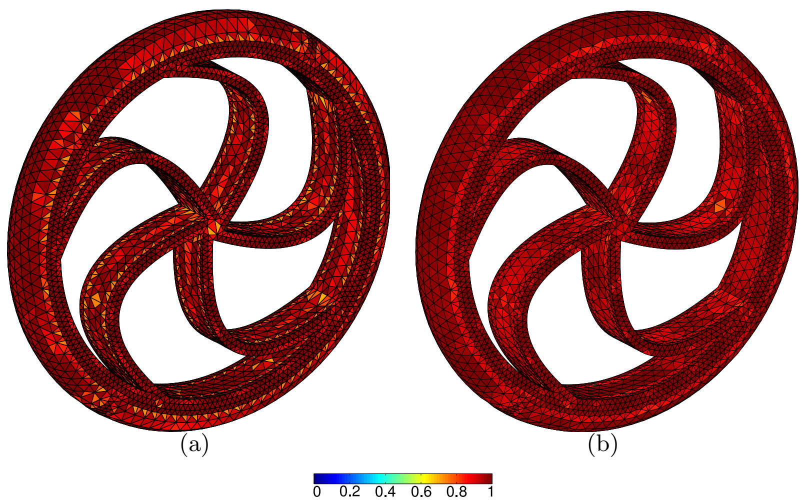

Fig. 6. Meshes for a component of a gear box colored according to the shape quality measure: (a) initial mesh, and (b) smoothed mesh.

Table 3. Shape quality statistics of the meshes on the component of a gear box and a crank shaft.

Surface	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Comp. gear	Initial	6(a)	0.56	1.00	0.94	0.07	0
	Smoothed	6(b)	0.74	1.00	0.95	0.04	0
Crank shaft	Initial	7(a)	0.49	1.00	0.92	0.09	0
	Smoothed	7(b)	0.59	1.00	0.93	0.06	0

shaft. Figure 6(a) shows the initial mesh on the component of the gear box. It is composed by 5723 nodes and 11462 triangular elements. Figure 6(b) presents the smoothed mesh. Then, Figure 7(a) presents the initial mesh on the crank shaft. It is composed by 4664 nodes and 9328 triangular elements. Figure 7(b) presents the resulting mesh from the smoothing procedure.

Table 3 details the shape quality statistics of the presented meshes. Note that the smoothing procedure properly improves the quality of the surface mesh in both cases. Moreover, it increases the minimum and the mean value of the quality of the mesh.

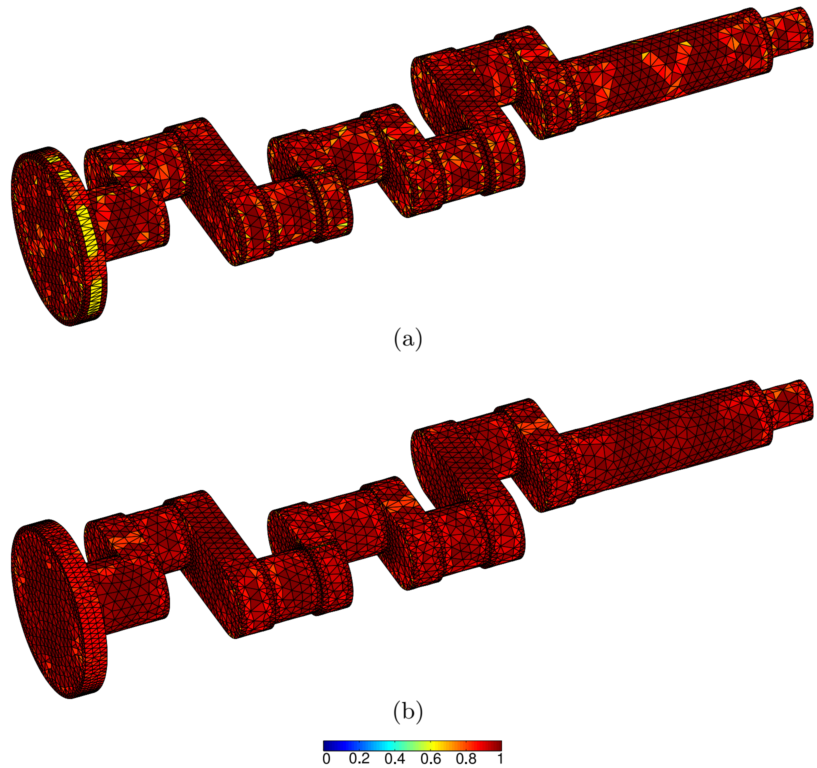

Fig. 7. Triangular meshes for a crank shaft colored according to the shape quality measure: (a) initial mesh, and (b) smoothed mesh.

6.4 Measures for high-order surface meshes

We present two applications of the extended measure for high-order elements on parameterized surfaces. To this end, we extend the optimization procedure presented in Section 4 for linear elements, to the measure for high-order elements presented in Equation 4. For both examples we use the shape distortion measure. The goal of the first example is to illustrate that the extended measure allows smoothing and untangling high-order meshes with a large number of tangled elements. We consider a revolution surface meshed with triangular elements, see Figure 8. Figure 8(a) presents a tangled mesh composed by a total of 522 triangle elements of order 2 (1121 nodes). Figure 8(b) presents the smoothed and untangled mesh. Next, we generate a mesh of order 5 composed by 7558 nodes and 532 triangle elements on a propeller. Figure 9(a) shows the initial high-order mesh, where the edges correspond to straight segments on the parametric space. Figure 9(b) presents the mesh resulting from the optimization procedure using the shape distortion measure. Table 4 presents the shape quality statistics corresponding to the presented high-order meshes.

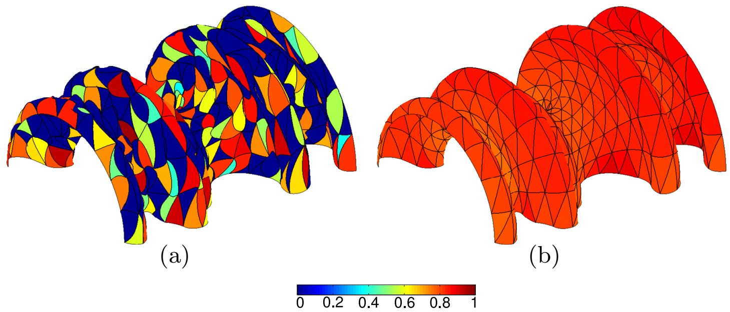

Fig. 8. Second order triangular meshes colored according to the shape quality measure for a revolution surface: (a) tangled mesh, and (b) smoothed and untangled mesh.

Table 4. Shape quality statistics of the meshes on a propeller.

Surface	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Revolution Srf	Initial	8(a)	0.00	0.96	0.79	0.30	297
	Smoothed	8(b)	0.75	0.90	0.81	0.03	0
Propeller	Initial	9(a)	0.22	0.97	0.79	0.13	0
	Smoothed	9(b)	0.46	0.97	0.81	0.10	0

7 Concluding remarks

In this work, we develop a new technique to define distortion (quality) measures for meshes on parameterized surfaces. Specifically, the proposed measures are expressed in terms of the parametric coordinates of the nodes. Then, we use the defined distortion measure to smooth and untangle meshes on parameterized surfaces. Since all the process is developed in the parametric space, the nodes are always on the CAD surface. In addition, we apply the proposed extension to define a distortion measure for high-order meshes on parameterized surfaces. We show the applicability of the optimization technique for two different planar distortion measures. In addition, we show that the method can be used to smooth and untangle linear and high-order meshes on parameterized surfaces.

Acknowledgement. Work partially sponsored by the Spanish *Ministerio de Ciencia e Innovación* (grant DPI2011-23058), and by CUR from DIUE of the *Generalitat de Catalunya* and the European Social Fund (grants FI-DGR and BE-DGR).

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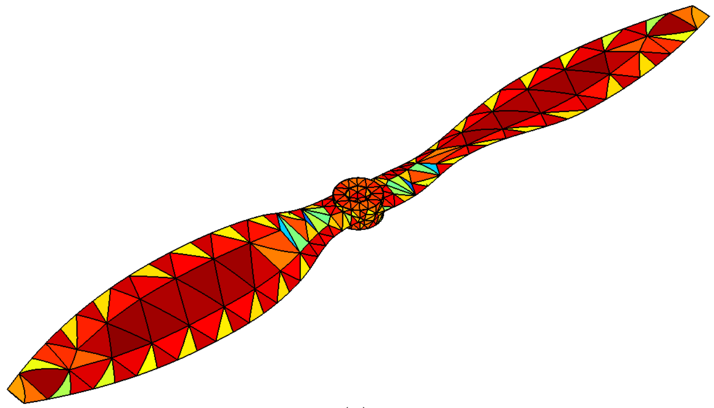
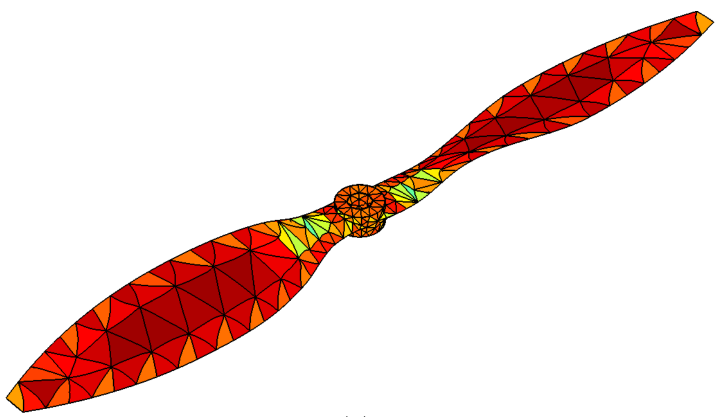


(a)



(b)

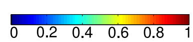


Fig. 9. Triangular meshes of order 5 colored according to the shape quality measure on a propeller: (a) initial mesh, and (b) smoothed mesh.

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