Decomposition of Domains into Quad Blocks Using the Medial Axis Transform

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Summary. This paper describes a novel approach to generating a multi-block decomposition of 2-D domains into high quality quad sub-regions using the medial axis transform (MAT). Bunin's continuum theory for unstructured mesh generation [1] is used as a means of understanding how the information contained in the MAT can be used to find effective positions for mesh dislocations.

Key Words. Multi-block decomposition, mesh dislocation positioning, medial axis transform

1 Introduction

The objective of this research is to develop an algorithm which can decompose an arbitrarily shaped 2-D geometry into well-shaped structured mesh mappable quad blocks. Dividing complex domains into such sub-regions involves considering the inherent geometric properties of the boundaries. Typically, it is imperative that meshes have high mesh quality on the boundaries. Thus mesh rows should follow the paths of smooth boundary edges and suitable mesh behaviour should occur at sharp boundary corners depending on the corner angle value. Once the mesh behaviour has been decided over all boundary elements only certain types of internal mesh flow are possible, or equivalently, the minimum number of internal mesh dislocations is fixed. The locations and number of dislocations (must be more than the minimum amount) needs to be decided and this choice dictates the overall shape of the mesh. A logical objective is to ensure that mesh flow lines (or *dual chords*) are as straight as possible.

2 Preliminaries

2.1 Medial Axis Transform

The *medial axis* of a domain is defined as the locus of the centers of the circles of maximal radius that can be inscribed in the domain [2]. Alternatively, it can be thought of as the set of all points having more than one nearest point on the boundary [3]. Edges and vertices in the medial axis are referred to as medial edges and medial vertices respectively.

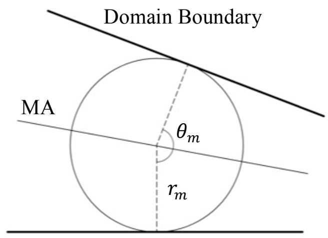

Fig. 1. Medial axis terminology

The medial axis together with the associated radius function (r_m) is called the *medial axis transform* (MAT) and contains all the information needed to reconstruct the original domain. The MAT is a useful description of the domain for decomposition purposes because it reveals which edges are proximate and the distance between them. Another useful piece of information is the angle between the maximal circle's center point and the touching points, which is termed here as the medial angle (θ_m).

2.2 Quadrilateral Mesh Dislocations

In unstructured quad meshes, irregularity points or *dislocations* are at nodes where more or less than the regular number of cells meet. Figure 2 and Figure 3 exemplify internal and boundary dislocations.

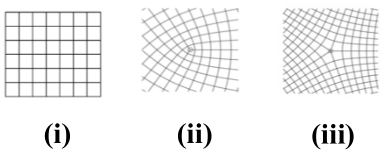

Fig. 2. Internal mesh dislocations: (i) Structured mesh ($k = 0$), (ii) Unstructured mesh with negative dislocation ($k = -1$), (ii) Unstructured mesh with positive dislocation ($k = 1$) (Taken from [1])

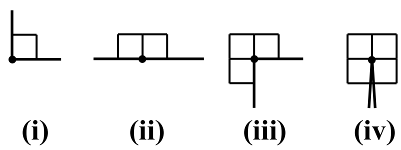

Fig. 3. Boundary Dislocations: (i) negative dislocation ($k_b = -1$), (ii) zero dislocation ($k_b = 0$), (iii) positive dislocation ($k_b = 1$), (iv) double positive dislocation ($k_b = 2$)

Appropriate boundary dislocations are needed at geometry corners to keep the skewness of the cells small. Internal dislocations are undesirable because the cells at the dislocation are inevitably skewed. However, a certain minimum number of internal dislocations are required depending on the geometry of the domain. Usually only simple internal dislocations $(k = \pm 1)$ should be used because higher degree dislocations result in poorer quality cells.

2.3 Mathematical Description of Unstructured Quadrilateral Meshes

An approach to high quality unstructured quadrilateral mesh generation described in [1] relates the problem of generating a quad mesh, where in the limit of increasing refinement cells become squares, to the problem of finding a conformally equivalent flat metric to the metric of the surface. In Riemannian manifold terminology:

$$\underbrace{\tilde{g}_{ij}}_{\text{New metric}} = \underbrace{e^{2\phi}}_{\text{conformal factor}} \underbrace{g_{ij}}_{\text{Old metric}} \qquad \underbrace{K \neq 0}_{\text{Old metric not generally locally flat}} \longrightarrow \underbrace{\widetilde{K} = 0}_{\text{New metric locally flat}} \tag{1}$$

If a square grid is drawn on the new metric $\tilde{g}_{ij}$, mesh edges lines or *mesh flow lines* are geodesics of $\tilde{g}_{ij}$. The scalar field ϕ must obey certain mathematical conditions where ϕ is the potential variable in a potential theory problem and the domain is subjected to flux loadings along boundaries and point source loadings acting at dislocations. A valid solution fully describes how mesh flow lines behave in the domain.

3 Placement of Dislocations

With the objective of trying to keep the paths of mesh flow lines as straight as possible through the domain, the optimum boundary dislocations at boundary corners and the positions of internal dislocations can be deduced using the mathematical theory in [1]. Curvature of mesh flow lines is caused by flux normal to the direction of travel of mesh flow lines and therefore the total flux between proximate boundary edges should be kept as low as possible.

The corner angle ranges for optimum boundary dislocation types given in Table 1 are found by minimising the flux between the involved boundaries. A single negative rather than a double negative boundary dislocation should probably be used for very acute corners (Table 2). Also, the preferred mesh flow between proximate boundaries depending on the medial angle can be found by minimising the flux between the boundaries linked by the medial axis, which is illustrated in Figure 5.

Table 1. Corner Angles Ranges for Boundary Dislocations

Range	k_b
$0 < \theta_{in} < \frac{\pi}{4}$	-2
$\frac{\pi}{4} < \theta_{in} < \frac{3\pi}{4}$	-1
$\frac{3\pi}{4} < \theta_{in} < \frac{5\pi}{4}$	0
$\frac{5\pi}{4} < \theta_{in} < \frac{7\pi}{4}$	1
$\frac{7\pi}{4} < \theta_{in} < 2\pi$	2

Table 2. Very Acute Corner Boundary Dislocation

Range	k_b
$0 < \theta_{in} < \frac{\pi}{4}$	-1

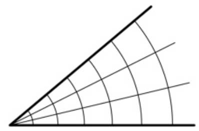

Fig. 4. Double negative boundary dislocation ($k_b = -2$)

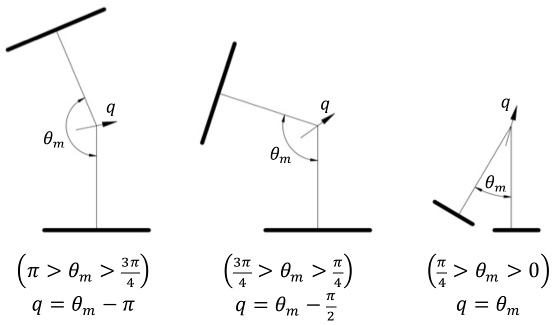

Fig. 5. Relationship between optimum net heat flow,q, through medial radii and the medial angle, θ_m.

Internal dislocations are needed at critical locations on the medial axis where there is a switch in the preferred mesh flow behaviour. This occurs at the critical medial angles of $\theta_m = 3\pi/4$ and $\theta_m = \pi/4$. Dislocations are also needed at medial vertices where the mesh flow behaviours associated with the medial edges are incompatible. In these instances there is a net flux into the location with a value of $n\,\pi/2$.

4 Examples

Figures 6 and 7 give a pictorial illustration of the decomposition process for two arbitrarily shaped 2-D simply connected domains with concave corners. Dislocations are found on the medial axis using the medial angle information, a potential theory problem with suitable boundary conditions is set up and solved numerically, and the full decomposition is generated using the ϕ-field solution.

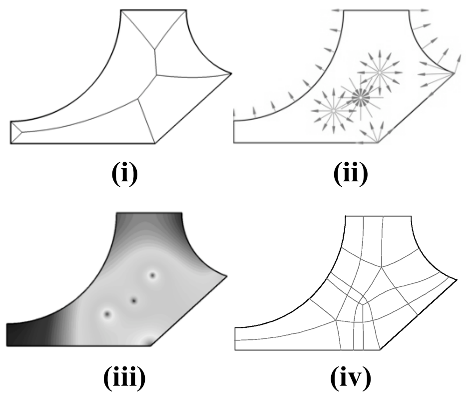

Fig. 6.

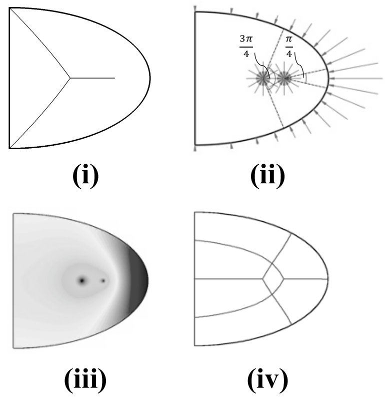

Fig. 7.

(i) Medial axis, (ii) Neumann boundary conditions and point source loadings at dislocations, (iii) Numerical solution of ϕ-field, (iv) Quad block decomposition of the domain

5 Conclusion

A method for finding an effective arrangement of mesh dislocations using the MAT has been devised. The decomposition edges are currently generated using a PDE numerical solution in the domain but a routine for obtaining a multi-block decomposition using only the geometric information contained in the MAT should be possible with this approach. The mesh flow in all regions of the domain is decided in the dislocation positioning step and hence the appearance of a quad mesh has been set. This is the current research focus.

References

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