Model Sensitivity of Edges to a Parameter

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Geometry differentiation is necessary when using solid model representations within design frameworks that employ gradient-based optimization. This prerequisite becomes a greater challenge if the solid model was obtained from a Computer Aided Design (CAD) system. In these cases, the solid model is usually a manifold boundary representation (BRep) created from a *master model* that contains driving parameters and a three-dimensional (3D) feature-construction recipe. Since access to the CAD system source code is normally unavailable and a geometry differentiation capability is also unavailable within the CAD system, other methods of differentiation must be employed.

With access to the master-model, parameters driving the solid model can be perturbed by some step-size in order to determine the design velocity (i.e. the mapping of a geometry feature from an initial to a perturbed domain) of each feature. This translates into a design velocity for each face, trim curve and node in the BRep. Armstrong et al. [1] utilized finite-differencing to find the design velocity normal to the model boundary in this manner. Although doing this for each driving parameter and topology feature in a complex model is tedious (requiring at least one new instance for each parameter perturbation), others have attempted different approaches. For example, to approximate geometry differentiation (assuming no topology changes) for aerodynamics analysis of an aircraft outer mold-line, Nemec et al. [2, 3] used centered finite-differencing of surface tessellations while tracking mesh nodes to determine the design velocity w.r.t. model parameters. Chen et al. proposed a method in [4] that required deriving implicit representations of active primitives in a solid model to determine the design velocity of active boundaries driven by model parameters.

In this research note, a method is presented for calculating the design velocity of trim curves, or edges, w.r.t. model parameters that drive the intersecting surfaces. As noted in some texts, such as [5], the design velocity along the intersection of parametrically defined surfaces can be determined using a vector equation (obtained by noting that the intersection curve is common to both surfaces) and an additional scalar constraint equation; however, the added constraint equation is not given. Herein a rigorous derivation of design velocity for an edge is presented by specifying a reasonable constraint equation and solving for the sensitivity to model parameters.

Derivation of the Sensitivity Along an Edge

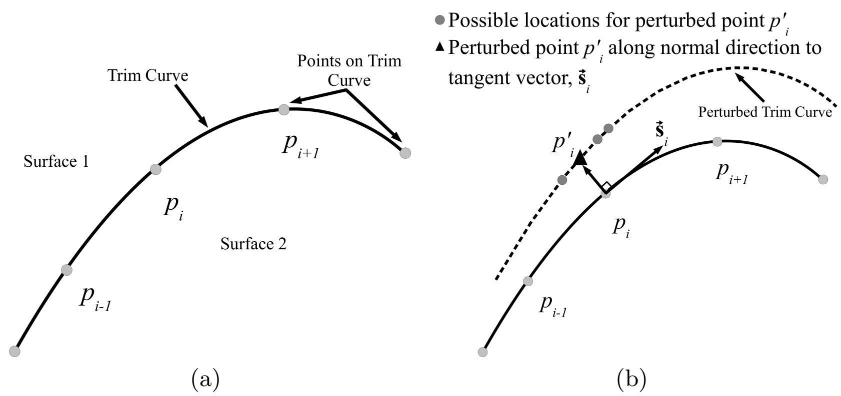

Fig. 1. (a) A trim curve defines the intersection of two surfaces. The point of interest on the trim curve is labeled p_i. **(b)** Perturbation of the trim curve and the resulting movement of p_i as viewed normal to the local tangent vector

Let two distinct surfaces in $\Re^3$, named surface 1 and 2, that are at least $\mathcal{C}^1$-regular share a common edge and be defined via the coordinates (u_1, v_1) and (u_2, v_2), respectively. These surfaces are parameterized by parameters $\{P_1, P_2 \dots P_j \dots P_J\}$, where each parameter P_j influences either surface 1, or surface 2, or both. A point $p = [x, y, z]^T$ (in Euclidean space) on surface 1 can be represented as $\vec{\mathbf{r}}_1(u_1, v_1; P_j)$, and on surface 2 a point can be described as $\vec{\mathbf{r}}_2(u_2, v_2; P_j)$.

The intersection of surfaces 1 and 2 is represented as a trim curve, $\vec{\mathbf{r}}_T$, shown in Figure 1(a), where the vector equation $\vec{\mathbf{r}}_{1_i} - \vec{\mathbf{r}}_{2_i} = 0$ is satisfied at each p_i along $\vec{\mathbf{r}}_T$. When one or more of the parameters P_j are perturbed, surfaces 1 and 2 are perturbed as well by some $\delta\vec{\mathbf{r}}_1$ and $\delta\vec{\mathbf{r}}_2$, respectively. This results in a perturbation of the trim curve to $\vec{\mathbf{r}}'_T$. Figure 1(b) illustrates a perturbed trim curve and possible locations for the perturbed point p'_i. At any p'_i along $\vec{\mathbf{r}}'_T$, we can then write

$$
\begin{aligned}
\vec{\mathbf{r}}'_1 - \vec{\mathbf{r}}'_2 &= 0 \\
\vec{\mathbf{r}}_1 + \delta\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 - \delta\vec{\mathbf{r}}_2 &= 0 \\
(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2) + (\delta\vec{\mathbf{r}}_1 - \delta\vec{\mathbf{r}}_2) &= 0 \\
\Rightarrow \delta\vec{\mathbf{r}}_1 - \delta\vec{\mathbf{r}}_2 &= 0 \qquad (1)
\end{aligned}
$$

Both $\delta\vec{\mathbf{r}}_1$ and $\delta\vec{\mathbf{r}}_2$ can be expressed by the linearized responses of $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ to the parameter perturbations δP_j at p_i, giving

$$\delta\vec{\mathbf{r}}_1 = \frac{\partial\vec{\mathbf{r}}_1}{\partial u_1}\frac{\partial u_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial v_1}\frac{\partial v_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial P_j}\delta P_j \tag{2}$$

$$\delta\vec{\mathbf{r}}_2 = \frac{\partial\vec{\mathbf{r}}_2}{\partial u_2}\frac{\partial u_2}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_2}{\partial v_2}\frac{\partial v_2}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_2}{\partial P_j}\delta P_j. \tag{3}$$

Note that summation over the parameters P_j is implied in (2) and (3), meaning

$$\delta\vec{\mathbf{r_1}} = \frac{\partial\vec{\mathbf{r}}}{\partial u_1}\sum_j \frac{\partial u_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}}{\partial v_1}\sum_j \frac{\partial v_1}{\partial P_j}\delta P_j + \sum_j \frac{\partial\vec{\mathbf{r}}}{\partial P_j}\delta P_j$$

$$\delta\vec{\mathbf{r_2}} = \frac{\partial\vec{\mathbf{r}}}{\partial u_2}\sum_j \frac{\partial u_2}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}}{\partial v_2}\sum_j \frac{\partial v_2}{\partial P_j}\delta P_j + \sum_j \frac{\partial\vec{\mathbf{r}}}{\partial P_j}\delta P_j.$$

Equations (2) and (3) are then substituted into (1) to yield

$$\frac{\partial\vec{\mathbf{r}}_1}{\partial u_1}\frac{\partial u_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial v_1}\frac{\partial v_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial P_j}\delta P_j - \left[\frac{\partial\vec{\mathbf{r}}_2}{\partial u_2}\frac{\partial u_2}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_2}{\partial v_2}\frac{\partial v_2}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_2}{\partial P_j}\delta P_j\right] = 0.$$

This equation must hold for any set of chosen δP_j's, therefore we require that the coefficient of each δP_j is zero:

$$\frac{\partial\vec{\mathbf{r}}_1}{\partial u_1}\frac{\partial u_1}{\partial P_j} + \frac{\partial\vec{\mathbf{r}}_1}{\partial v_1}\frac{\partial v_1}{\partial P_j} - \frac{\partial\vec{\mathbf{r}}_2}{\partial u_2}\frac{\partial u_2}{\partial P_j} - \frac{\partial\vec{\mathbf{r}}_2}{\partial v_2}\frac{\partial v_2}{\partial P_j} = \frac{\partial\vec{\mathbf{r}}_2}{\partial P_j} - \frac{\partial\vec{\mathbf{r}}_1}{\partial P_j}. \tag{4}$$

Equation (4) is a vector equation containing the three $[x, y, z]$ component equations. Since the unknowns we desire to compute are $\frac{\partial u_1}{\partial P_j}$, $\frac{\partial v_1}{\partial P_j}$, $\frac{\partial u_2}{\partial P_j}$ and $\frac{\partial v_2}{\partial P_j}$, a fourth equation is needed. The chosen fourth equation also needs to represent the trim curve that each p_i pertains to. One way of identifying this added relation is by using the tangent vector, $\vec{\mathbf{s}}$, at p_i on the trim curve $\vec{\mathbf{r}}_T$. We choose to use a perturbation $\delta\vec{\mathbf{r}}_T$ that lies normal to $\vec{\mathbf{s}}$ to create some point p_i', which gives $\delta\vec{\mathbf{r}}_T \cdot \vec{\mathbf{s}} = 0$. Since $\vec{\mathbf{r}}_T = \vec{\mathbf{r}}_1$, we can substitute (2) into this fourth equation and obtain the scalar equation

$$\left(\frac{\partial\vec{\mathbf{r}}_1}{\partial u_1}\frac{\partial u_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial v_1}\frac{\partial v_1}{\partial P_j}\delta P_j + \frac{\partial\vec{\mathbf{r}}_1}{\partial P_j}\delta P_j\right)\cdot\vec{\mathbf{s}} = 0$$

$$\Rightarrow \left(\frac{\partial\vec{\mathbf{r}}_1}{\partial u_1}\cdot\vec{\mathbf{s}}\right)\frac{\partial u_1}{\partial P_j} + \left(\frac{\partial\vec{\mathbf{r}}_1}{\partial v_1}\cdot\vec{\mathbf{s}}\right)\frac{\partial v_1}{\partial P_j} = -\frac{\partial\vec{\mathbf{r}}_1}{\partial P_j}\cdot\vec{\mathbf{s}}. \tag{5}$$

At this point, the sensitivity of point p_i on the trim curve $\vec{\mathbf{r}}_T$ to parameters P_j can be determined by combining (4) and (5) into

$$\begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial v_1} & -\frac{\partial x_2}{\partial u_2} & -\frac{\partial x_2}{\partial v_2} \\ \frac{\partial y_1}{\partial u_1} & \frac{\partial y_1}{\partial v_1} & -\frac{\partial y_2}{\partial u_2} & -\frac{\partial y_2}{\partial v_2} \\ \frac{\partial z_1}{\partial u_1} & \frac{\partial z_1}{\partial v_1} & -\frac{\partial z_2}{\partial u_2} & -\frac{\partial z_2}{\partial v_2} \\ \frac{\partial \vec{\mathbf{r}}_1}{\partial u_1}\cdot\vec{\mathbf{s}} & \frac{\partial \vec{\mathbf{r}}_1}{\partial v_1}\cdot\vec{\mathbf{s}} & 0 & 0 \end{bmatrix}_i \begin{bmatrix} \frac{\partial u_1}{\partial P_j} \\ \frac{\partial v_1}{\partial P_j} \\ \frac{\partial u_2}{\partial P_j} \\ \frac{\partial v_2}{\partial P_j} \end{bmatrix}_i = \begin{bmatrix} \frac{\partial x_2}{\partial P_j} - \frac{\partial x_1}{\partial P_j} \\ \frac{\partial y_2}{\partial P_j} - \frac{\partial y_1}{\partial P_j} \\ \frac{\partial z_2}{\partial P_j} - \frac{\partial z_1}{\partial P_j} \\ -\frac{\partial \vec{\mathbf{r}}_1}{\partial P_j}\cdot\vec{\mathbf{s}} \end{bmatrix}_i , \qquad (6)$$

which is a 4×4 linear system with multiple righthand sides, one for each parameter P_j. The coefficient matrix and the righthand sides are readily evaluated from CAD data. It is understood that if $\vec{\mathbf{r}}_1$ does not depend on one particular P_j, then

$$\frac{\partial \vec{\mathbf{r}}}{\partial P_j} = \frac{\partial x_1}{\partial P_j} = \frac{\partial y_1}{\partial P_j} = \frac{\partial z_1}{\partial P_j} = 0$$

in the j'th righthand side expression, and likewise for $\vec{\mathbf{r}}_2$.

The system (6) is readily solved using pivoting LU decomposition and multiple back-substitutions, which yields the sensitivities of u_1, v_1, u_2, v_2 w.r.t. P_j at point p_i on the trim curve:

$$\Rightarrow \qquad \begin{bmatrix} \frac{\partial u_1}{\partial P_j} \\ \frac{\partial v_1}{\partial P_j} \\ \frac{\partial u_2}{\partial P_j} \\ \frac{\partial v_2}{\partial P_j} \end{bmatrix}_i \qquad , \qquad j = 1 \dots J \qquad (7)$$

These derivatives can then be used in (2) or (3) to compute the linear displacement of point p_i in response to any set of chosen δP_j's. All of these operations are repeated for each point p_i on the trim curve, thus allowing the calculation of the overall trim curve's linear displacement in response to the δP_j's.

References

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