A New Anisotropic Mesh Adaptation Method Based upon Hierarchical A Posteriori Error Estimates

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Summary. An anisotropic mesh adaptation strategy for finite element solution of elliptic differential equations is considered. The adaptation method generates anisotropic adaptive meshes as quasi-uniform ones in some metric space. The associated metric tensor is computed by means of a posteriori hierarchical error estimates. A global hierarchical error estimate is employed in this study to obtain reliable directional information of the solution. Mesh examples are presented for the mathematical model for heat conduction in a thermal battery with large orthotropic jumps in the material coefficients[4].

Key words: mesh adaptation; anisotropic mesh; finite element; a posteriori error estimates

1 Introduction

Anisotropic mesh adaptation has proved to be a useful tool in numerical solution of partial differential equations. This is especially true when problems arising from science and engineering have distinct anisotropic features. The ability to adapt the size, shape, and orientation of mesh elements according to certain quantities of interest can significantly improve the accuracy of the solution and enhance the computational efficiency.

A common approach for generating an anisotropic mesh is based on generation of a quasi-uniform mesh in some metric space. Typically, the appropriate metric depends on the Hessian of the exact solution of the underlying problem, which is often unavailable in practical computation. The common approach to avoid this difficulty is to recover an approximate Hessian from the computed solution. The purpose of this research is to consider an alternative approach and to study the use of a posteriori error estimates in anisotropic mesh adaptation.

[4] A Sandia National Laboratories benchmark problem.

A key idea in the new approach is the use of the hierarchical error estimator for reliable directional information of the solution. The solution error can be bounded by the interpolation error of an appropriately defined reconstruction applied to the finite element approximation. One possibility to achieve the desired property is to use the hierarchical decomposition of the finite element space. The solution error is then bounded by the explicitly computable interpolation error of the hierarchical basis error estimator. Anisotropic interpolation error estimates are developed in [2, 3] and [4]. We follow the theory in [4] to define the optimal metric tensor for minimizing the interpolation error of the hierarchical a posteriori error estimator.

It has been pointed out that error estimation based on solving local error problems can be inaccurate on anisotropic meshes [1]. We thus choose to develop our approach based on error estimation by means of globally defined error problem. To avoid the expensive exact solution of the global error problem, we employed only a few steps of the symmetric Gauß-Seidel iteration for the efficient solution of the resulting linear system. Numerical results have shown that this is sufficient for obtaining an approximation to the error good enough for the purpose of mesh adaptation.

First numerical results have shown that the new method is fully comparable in accuracy with commonly used Hessian-recovery-based methods and can be more efficient for some examples by producing only necessary element concentration.

2 Numerical Example: Heat Conduction in a Thermal Battery

We consider heat conduction in a thermal battery with large orthotropic jumps in the material coefficients. The mathematical model considered here is taken from [5, 6] and described by

$$\begin{cases} \nabla \cdot (D^k \nabla u) = f^k & \text{in } \Omega, \\ D^k \nabla u \cdot n = g^i - \alpha^i u & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega = (0, 8.4) \times (0, 24)$ and

$$D^k = \begin{bmatrix} D_x^k & 0 \\ 0 & D_y^k \end{bmatrix}.$$

The data for each material k and for each of the four sides i of the boundary starting with the left hand side boundary and ordering them clockwise are given in Table 1 .

The analytical solution for this problem is unavailable. The geometry and the contour and surface plots of a finite element approximation are given in Fig. 1. Typical adaptive meshes with predefined interface edges are shown in Fig. 2.

When the mesh contains all the information of the interface, common recovery-based adaptation methods will produce a mesh with strong element concentration near all internal interfaces (Fig. 2a, using quadratic least squares Hessian recovery as proposed in [7]), whereas the error estimator leads to a mesh that has higher element concentration in the corners of the regions (Fig. 2b), has a proper element orientation near the interfaces between the regions 2 and 3, and is almost uniform in regions where the solution is nearly linear (cf. Fig. 1c for the surface plot of a computed solution).

Thus, the new method produces only necessary concentration and is able to catch the directional information of the solution required for proper element alignment. This example demonstrates that the new method can be successfully used for problems with strong anisotropic features and jumping coefficients.

References

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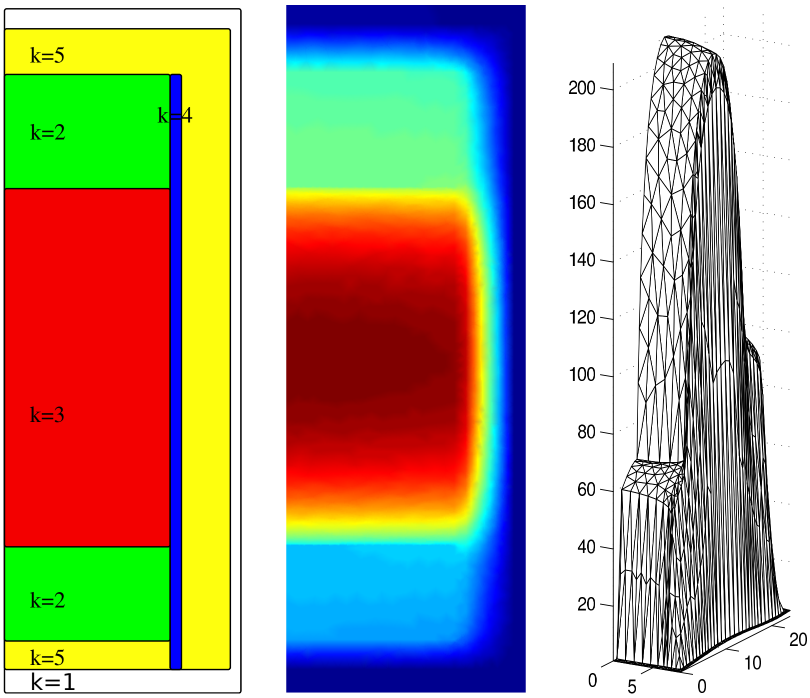
Region k	D_x^k	D_y^k	f^k
1	25	25	0
2	7	0.8	1
3	5	0.0001	1
4	0.2	0.2	0
5	0.05	0.05	0

(a) Material coefficients.

Boundary i	α^i	g^i
1	0	0
2	1	3
3	2	2
4	3	0

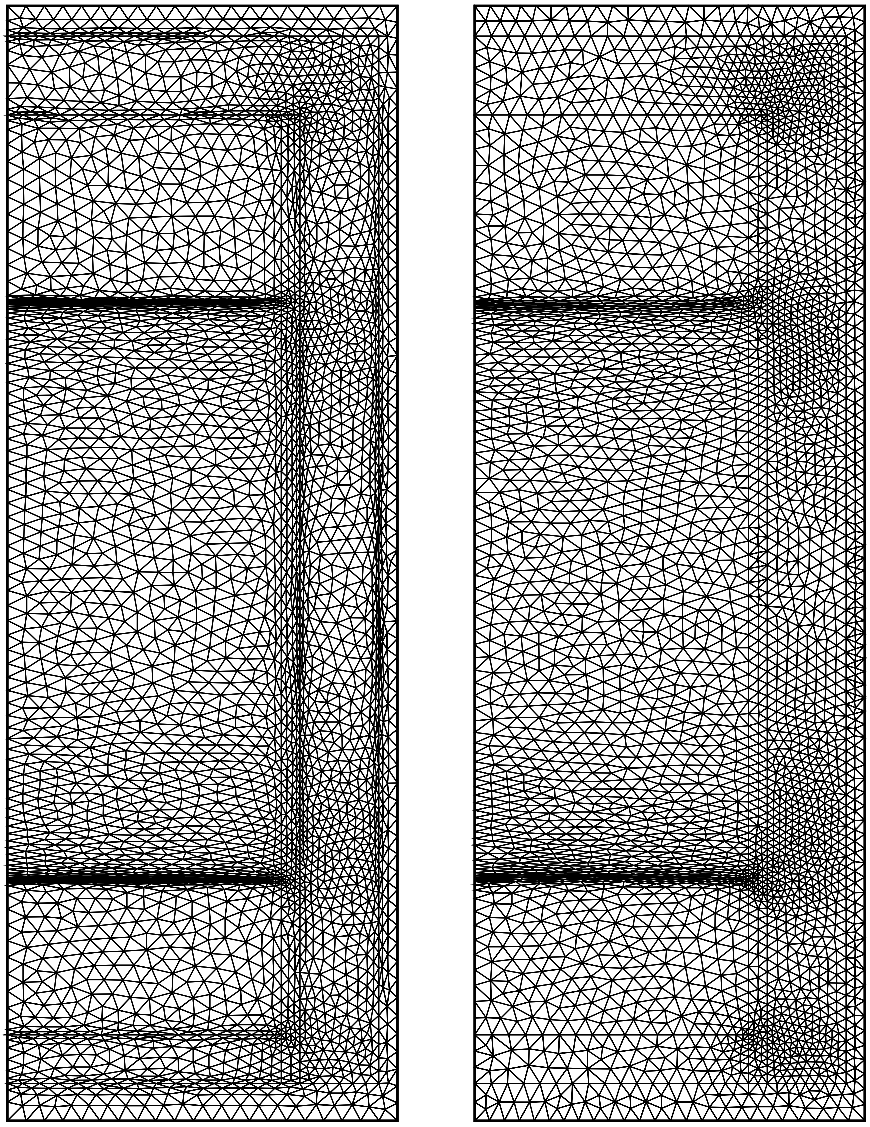
(b) Boundary conditions.

Table 1: Heat conduction in a thermal battery: material coefficients and boundary conditions.



(a) (b) (c)

Fig. 1: Heat conduction in a thermal battery: (a) device geometry, (b) contour plot, and (c) surface plot of a linear finite element solution.



(a) Quadratic least squares Hessian recovery: maximum aspect ratio 57.3.

(b) Error estimator: maximum aspect ratio 60.8.

Fig. 2: Heat conduction in a thermal battery: adaptive meshes obtained by means of (a) quadratic least squares Hessian recovery and (b) hierarchical basis a posteriori error estimator.