# A New Set of Hexahedral Meshes Local Transformations

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**Abstract.** The modification of hexahedral meshes is difficult to perform since their structure does not allow easy local refinement or un-refinement such that the modification does not go through the boundary. In this paper we prove that the set of hex flipping transformations of Bern et. al. [1] is the only possible local modification on a geometrical hex mesh with less than 5 edges per vertex. We propose a new basis of local transformations that can generate an infinite number of transformations on hex meshes with less than 6 edges per vertex. Those results are a continuation of a previous work [9], on topological modification of hexahedral meshes. We prove that one necessary condition for filling the enclosed volume of a surface quad mesh with compatible hexes is that the number of vertices of that quad mesh with 3 edges should be no less than 8. For quad meshes, we show the equivalence between modifying locally the number of quads on a mesh and the number of its internal vertices.

## 1 Introduction

The using of hexahedral or quadrilateral meshes has its advantages. For example, the smaller number of elements obtained for a given number of vertices than tetrahedral or triangle meshes. But one difficulty within hexahedral meshes is to transform it locally without touching a large region of the mesh, due to its structure as set of layers. In [1] Bern et. al. propose the set of hex flipping transformations as local transformations of hex meshes i.e., the modifications does not change the boundary quads of the local domain where they are applied.

In the second section, we prove that modifying locally quad meshes is equivalent to modifying the number of its internal vertices. In the third section, we propose a necessary condition for meshing a volume with hexes from its boundary quad mesh. We study the problem of local transformations on hex meshes with less than 5 edges per vertex. That brings us to the next section where we look at the hex flipping transformations of Bern et. al. as the unique set of local transformations on that type of hex meshes. In the fifth section, we extend those local transformations to hex meshes with less than 6 edges per vertex and propose a new set of geometrical hex local transformations.

### 1.1 Definitions and Notations

A *polyhedron* is a set of closed planar polygonal faces in $\mathbb{R}^3$ that intersect only at shared vertices and edges, and such that each edge is shared by exactly two faces and each vertex is shared by exactly one cycle of faces.[1] A polyhedron divides $\mathbb{R}^3$ into a bounded *interior* and an unbounded *exterior*, each of which may contain more than one connected component.

A *quad surface mesh* is a polyhedron whose faces are all quads. A *hex mesh* is a quad surface mesh, along with interior quads that subdivide the polyhedron's interior into hexes. The intersection of any pair of hexes is either empty, a single vertex, a single edge, or a single quad. Quad surface meshes and hex meshes are both examples of *cubical complexes* [2, 8, 10], the analogue of simplicial complexes in which each $k$-dimensional face is a $k$-dimensional cuboid.

In the following, if not mentioned, we'll consider only *non singular* simply connected meshes (without holes): simply connected meshes where each element (quad or hex) has at least one common facet (edge for quad mesh or quad for hex mesh) with the rest of the mesh.

We call *elementary hex mesh* a conformal hex mesh without holes where every vertex has at the most 4 edges.

We call *interior* of a hex mesh the rest of the hex mesh obtained without its boundary vertices (boundary quads).

On a mesh, let us denote:

$P$ the number of Hexes (polyhedrons), $F$ the number of faces (quads),
$\bar{F}$ the number of boundary faces, $\mathring{F}$ the number of internal faces,
$E$ the number of edges, $\bar{E}$ the number of boundary edges,
$\mathring{E}$ the number of internal edges, $V$ the number of vertices,
$\mathring{V}$ the number of internal vertices, $\bar{V}$ the number of boundary vertices,
$V_j^k$ the number of vertices with $k$ boundary edges and $j$ internal edges.

We'll use Euler's and Cauchy's formulas [7, 3]:

$$F - E + V - P =, 1 \tag{1}$$

for a complex of polyhedrons, and

$$F - E + V = 2, \tag{2}$$

for a single polyhedron.

## 2 Quad Mesh and Local Modifications

In hexahedral and quadrilateral mesh generation, the existence of a mesh need some conditions such as an even number of facets on its boundary [5]; we give a generalization of this property to any mesh whose elements have an even number of boundary facets even if they are of different type.

[1] This definition of polyhedron allows multiple connected components (two cubes can be one polyhedron!), as well as voids and tunnels model objects made from two different materials.

**Theorem 1.** *Any conformal mesh whose elements (of the same type or not) have each an even number of facets, has an even number of facets on its boundary.*

*Proof.* Let us consider a mesh with $n$ elements.This is true for $n = 1$.
We suppose that this is the case for any mesh with $n$ elements. For a mesh with $n + 1$ elements, we have to add one element to some previous $n$-element mesh. So the added element intersects the boundary of the previous mesh at $m$ facets, $1 \leq m \leq k - 1$, if $k$ is the number of facets of the added element. If $b$ is the number of facets of the previous mesh, the boundary of the new mesh has $b - m + (k - m) = b + k - 2m$. Which is even.

It is known that quad meshes give a smaller number of elements than triangle meshes for a given set of vertices. Knowing the number of boundary and internal vertices, we have exactly the number of quads and edges in a quad mesh.

**Proposition 1.** *In a 2D simply-connected mesh with $h$ holes and $k$ vertices per element $(k \geq 3)$ the number $F$ of elements, $\bar{V}$ of boundary vertices, $\mathring{V}$ of internal vertices and $E$ of edges satisfies the relations:*

$$F = \frac{\bar{V} + 2(\mathring{V} + h - 1)}{k - 2}.$$

$$E = \frac{(k-1)\bar{V} + k(\mathring{V} + h - 1)}{k - 2}.$$

*Proof.* In a $2D$ mesh each internal edge $\mathring{E}$ belongs to two elements and each boundary edge $\bar{E}$ belongs to one element. So

$$2\mathring{E} + \bar{E} = kF.$$

As the boundary is a union of closed lines (only one if there are no holes), $\bar{E} = \bar{V}$.
Thanks to the Euler formula [3],

$$F + V - E = 2 - h.$$

A quad mesh is solution of the following equations:

$$\begin{cases} V - E + F = 2 - h \\ 2\mathring{E} + \bar{V} = kF \end{cases} \tag{3}$$

whose solutions give the formulas.

For quad meshes, to locally transform a mesh (i.e., a sub-domain of the mesh) when keeping the boundary of that sub-domain unchanged, amounts to moving only its internal vertices and re-meshing the sub-domain.

**Theorem 2.** *The variation of the number of quads equal the variation of the number of internal vertices between two quad meshes ($mesh_a$ and $mesh_b$), if and only if the two meshes have the same number of boundary vertices:*

$$\mathring{V}_a - \mathring{V}_b = F_a - F_b \iff \bar{V}_a = \bar{V}_b.$$

*In other words, adding or removing some elements in a quad mesh without changing its boundary is equivalent to adding or removing exactly the same number of its internal vertices.*

*Proof*

• Let us suppose that $\bar{V}_a = \bar{V}_b$. As we have said, on a quad mesh each internal edge is shared by exactly two quads so we can write:

$$4F = 2\mathring{E} + \bar{E}. \tag{4}$$

Thanks to the first equation of (3) the difference between two meshes that have the same boundary satisfies

$$(F_a - F_b) + (V_a - V_b) - (E_a - E_b) = 0,$$

so

$$(F_a - F_b) + (\mathring{V}_a - \mathring{V}_b) - (\mathring{E}_a - \mathring{E}_b) = 0.$$

Thanks to equation (4),

$$\mathring{E}_a - \mathring{E}_b = 2(F_a - F_b),$$

so

$$(\mathring{V}_a - \mathring{V}_b) - (F_a - F_b) = 0.$$

• Let us suppose that $\mathring{V}_a - \mathring{V}_b = F_a - F_b$. Then

$$(F_a - F_b) + (V_a - V_b) - (E_a - E_b) = 0$$

gives

$$2(\mathring{V}_a - \mathring{V}_b) + (\bar{V}_a - \bar{V}_b) - (E_a - E_b) = 0.$$

Equation (4) gives $4(\mathring{V}_a - \mathring{V}_b) = 2(\mathring{E}_a - \mathring{E}_b) + (\bar{E}_a - \bar{E}_b)$, so

$$2(\bar{E}_a - \bar{E}_b) - 2(\bar{V}_a - \bar{V}_b) = \bar{E}_a - \bar{E}_b.$$

But on each quad mesh, $\bar{E} = \bar{V}$ so

$$\bar{E}_a = \bar{E}_b.$$

## 3 Hexahedral Meshing and Local Modifications

As a complex of polyhedrons, a hex mesh satisfies the equation (1). Its boundary is a polyhedron so it satisfies (2). On its boundary, each edge belongs exactly to two quads and the number of hex times the number of faces per hex equals the number of boundary faces plus two times the number of internal faces. Then any hes mesh is subject to the following system:

$$\begin{cases} V - E + F - P = 1 \\ \bar{V} - \bar{E} + \bar{F} = 2 \\ \bar{E} = 2\bar{F} \\ \bar{F} + 2\mathring{F} = 6P. \end{cases} \tag{5}$$

**Proposition 2.** *A simply-connected hex mesh satisfies the following equations:*

$$\bar{V} = \bar{F} + 2,$$

$$\sum_{j\geq 0,\ k\geq 3} (4-k)\, V_j^k = 8,$$

$$P = \frac{1}{2}\left(\mathring{E} - \mathring{V} - 1\right) + \frac{\bar{F}}{4}.$$

*Proof*

- The second equation of system (5) combined with the third gives

$$\bar{V} = \bar{F} + 2.$$

- The number of boundary edges satisfies $\sum_{j\geq 0,\ k\geq 3} kV_j^k = 2\bar{E} = 4\bar{F}$. As $\bar{F} = \bar{V} - 2$, we have

$$\sum_{j\geq 0,\ k\geq 3} (4-k)\, V_j^k = 8.$$

- The first minus the second equation of system (5) gives

$$\mathring{V} - \mathring{E} + \mathring{F} - P = -1,$$

so with $\mathring{F} = 3P - \frac{\bar{F}}{2}$, we obtain

$$P = \frac{1}{2}\left(\mathring{E} - \mathring{V} - 1\right) + \frac{\bar{F}}{4}.$$

Mitchell [11] gives some necessary conditions for meshing a domain with compatible hexes from an existing boundary quad mesh. This proposition gives another one.

**Theorem 3.** *The enclosed volume of a surface quad meshed with an even number of quads can be mesh with compatible hexes only if the number of vertices with 3 edges of that surface mesh is greater than or equal to 8.*

*Proof.* The second equation of the previous proposition gives

$$\sum_{j\geq 0} V_j^3 = 8 + \sum_{j\geq 0,\ k\geq 5} (k-4)\, V_j^k.$$

Since $k - 4 > 0\ \forall k \geq 5$, we have $\sum_{j\geq 0} V_j^3 \geq 8$.

### 3.1 Elementary Hex Meshes Local Modifications

Considering a hex mesh, we want to modify it locally such that in some region we modify a sub-mesh without changing the boundary. We also wish to investigate the changes of parity (i.e., a transformation between two configurations, one with an even number of hexes and the other with an odd number). As this is not an easy problem, we consider first a simple case.

We want to deal with the simplest (in terms of connectivity between vertices) sub-domain (sub-mesh) with more than one hex of any mesh. A mesh with only 3 edges per vertex is reduced to only one hex, so the simplest mesh should have some vertices with more than 3 edges. That is why we've define the *elementary hex mesh* and it implies some consequences.

**Lemma 1.** *The interior of an elementary hex mesh is a mesh (internal mesh) with at most one hex, one quad, one edge or one vertex.*

*Proof.* We suppose that the interior of an elementary hex mesh is a union of non connected (single) edges, vertices or faces. Then there is at least one boundary vertex sharing one edge with at least one single part of that interior, and one other edge with another boundary vertex or another single part of the interior of that elementary hex mesh. So that boundary vertex will have five edges which is not permitted. So, the interior of an elementary hex mesh has no single part; it is an *internal mesh* (1 dimension mesh, quad mesh or hex mesh).

If the interior of an elementary hex mesh is a mesh with more than one hex, there will be at least one boundary vertex of that internal mesh with four edges (a vertex shared by two hexes) plus the edge connecting that vertex to the boundary of that elementary hex mesh. This will give five edges, which is not admitted. If it has no hex but more than one quad, each face of those quads should belong to at least two different hexes (one per side). So each vertex of the intersection of two of those quads will have at least five edges. This is not admitted. If there is more than two aligned vertices, three for example, there will be at least two internal faces around the middle vertex which should be coplanar. So there is no way to have hexes above and below those faces by adding only one edge to that middle vertex.

As each internal vertex has 4 edges, we have

$$4(\mathring{V} + V_0^4 + V_1^3) + 3V_0^3 = 2E.$$

Each internal vertex belongs to exactly 4 hexes and to 6 faces, each boundary vertex with no internal edge belongs to two hexes and to five faces, each boundary vertex with one internal edge belongs to 3 hexes and to six faces and each boundary vertex with only 3 edges belongs to only one hex and to 3 faces. So we have

$$4\mathring{V} + V_0^3 + 2V_0^4 + 3V_1^3 = 8P,$$

$$6\mathring{V} + 3V_0^3 + 5V_0^4 + 6V_1^3 = 4F.$$

Euler's and Cauchy's formulas (1),(2) give:

$$F - E + V - P = 1,\ \bar{F} - \bar{E} + \bar{V} = 2.$$

Then

$$\mathring{F} - \mathring{E} + \mathring{V} - P = -1.$$

We can say that an elementary hex mesh is a solution of the following system

$$\begin{cases} 2\mathring{F} + \bar{F} = 6P \\ \mathring{F} - \mathring{E} + \mathring{V} - P = -1 \\ 4(\mathring{V} + V_0^4 + V_1^3) + 3V_0^3 = 2E \\ 4\mathring{V} + V_0^3 + 2V_0^4 + 3V_1^3 = 8P \\ 6\mathring{V} + 3V_0^3 + 5V_0^4 + 6V_1^3 = 4F \\ V_0^3 + V_0^4 + V_1^3 = 2 + \bar{F}. \end{cases} \tag{6}$$

It solution satisfies

$$\begin{cases} \mathring{F} = 3P - \frac{\bar{F}}{2} \\ \mathring{V} = 2P + 1 - \frac{\bar{F} + V_1^3}{2} \\ \mathring{E} = 4P + 2 - \frac{2\bar{F} + V_1^3}{2} \\ V_0^4 = \bar{F} - 6 \\ V_0^3 = 8 - V_1^3, \end{cases} \tag{7}$$

or

$$\begin{cases} \mathring{F} = \frac{6(\mathring{V} - 1) + \bar{F} + 3V_1^3}{4} \\ P = \frac{2(\mathring{V} - 1) + \bar{F} + V_1^3}{4} \\ \mathring{E} = 2\mathring{V} + \frac{V_1^3}{2} \\ V_0^4 = \bar{F} - 6 \\ V_0^3 = 8 - V_1^3. \end{cases} \tag{8}$$

**Theorem 4**

1. *No elementary hex mesh with more than 7 hexes can be transformed into another elementary hex mesh without changing its boundary quads.*
2. *The parity of an elementary hex mesh can't be changed without modifying its boundary quads.*

*Proof.* Let us suppose that we have an elementary hex mesh$_a$ that we want to transform into another elementary hex mesh$_b$ with the same boundary quads.

The number of transformations of any elementary hex mesh is the sum of the number of different ways to connect the boundary of that mesh to its internal mesh, each way times the number of possibles transformations of the interior of the internal mesh associated. This because if we modify the position of any boundary edge it will modify boundary quads. Thanks to the previous lemma, the interior of any internal mesh of an elementary hex mesh is empty (no vertex).

So the number of transformations is only the number of different possibilities to connect the boundary of the elementary hex mesh to each of its possible internal mesh. Let us examine how to modify the connection between the boundary and the interior of an elementary hex mesh.

Of course we have that $V_0^4 = \bar{F} - 6$ as part of the solution of the system (6), any transformation will keep the number of $V_0^4$ constant. The only possible modifications that can change the connection between the boundary and the interior, is to interchange $V_1^3$ and $V_0^3$. This means that any transformation will modify any boundary vertex of the first configuration (mesh$_a$) that has 3 edges to a boundary vertex of the second configuration (mesh$_b$) that has 3 boundary edges and one internal edge. If there is a boundary vertex that has 3 boundary edges not modified while the others are modified, the hexes that it belong to will not be transformed. So all the vertices that have 3 edges on the boundary will be modified by any transformation. Then there is at the most one possible transformation of the mesh$_a$ into the mesh$_b$ with the same boundary quads, which satisfies

$$\begin{cases} {V_0^3}_a = {V_1^3}_b \\ {V_1^3}_a = {V_0^3}_b \end{cases} \tag{9}$$

or

$$\begin{cases} {V_0^3}_a + {V_0^3}_b = 8 \\ {V_1^3}_a + {V_1^3}_b = 8 \end{cases}$$

Thanks to the previous lemma any internal mesh has at most one hex, so $\mathring{V} \leq 8$.

*Remark 1*

•${V_1^3}_a$ is always even because the number of edges going from the internal mesh (previous lemma) to the boundary is always even and when a boundary vertex of the mesh $a$ that has one internal edge is not connected to an internal vertex, it is connected to the same type of boundary vertex.

• Thanks to the solutions (8) of the system (6), we have

$$P_a - P_b = \frac{\mathring{V}_a - \mathring{V}_b + {V_1^3}_a - 4}{2}.$$

This means that $\mathring{V}_a - \mathring{V}_b$ have the same parity because ${V_1^3}_a$ is always even. Since $\bar{F}$ is always even (by theorem 1), let us look at the possible values of $\mathring{V}$ in the solutions (7).

- If $\mathring{V}_a = 8$,
  then ${V_1^3}_a = 8$ (each internal vertex will have one boundary edge). ${V_0^3}_a = 0, \mathring{V}_b = 0, P_a = 7$; one internal hex + six around its faces; since those 7 hexes have already 4 edges per vertex there is no way to add another one. In this case, we have $P_a - P_b = 6$, so $P_a + P_b = 8$.
- If $\mathring{V}_a = 4$,
  those internal vertices form one face around which there are 6 hexes: one above, one bellow and four around the four edges. Since each internal vertex

has four edges we will have ${V_1^3}_a = 8$ and $V_3^1 = 0$. We have ${V_1^3}_b = 0$ so $\mathring{V}_b = 0$. As there is no way to add another hex to the first six hexes, $P_a = 6$; we have $P_a - P_b = 4$ so $P_a + P_b = 8$.

- If $\mathring{V}_a = 2$,
  each internal vertex has four edges. We will have ${V_1^3}_a = 6$. Since $\mathring{V}_a$ and $\mathring{V}_b$ have the same parity, $\mathring{V}_b = 0$ or 2; the cases $\mathring{V} = 4$ and $\mathring{V} = 8$ have been examined above. If $\mathring{V}_b = 2$, then ${V_1^3}_b = 6$ which is absurd because of system (9) and the fact that ${V_1^3}_a = 6$. So $\mathring{V}_b = 0$ gives $P_a - P_b = 2$. For the same reasons as in the previous cases $P_a = 5$ (around 2 internal vertices there cannot be more or less than five hexes in an elementary hex mesh) then $P_a + P_b = 8$.
- If $\mathring{V}_a =, 1$
  then $\mathring{V}_b = 1$ (parity of both internal vertices); ${V_1^3}_a = {V_1^3}_b = 4$, $P_a = 4$, $P_a - P_b = 0$. With one internal edge, there is no way to have more or less than four hexes in an elementary hex mesh; so $P_a + P_b = 8$.
- If $\mathring{V}_a = 0$,
  We will examine only the case $\mathring{V}_b = 0$ because the other cases are contained in the previous ones. Let us examine the number of ${V_1^3}_a$:
  - If ${V_1^3}_a = 0$ we have $P_a - P_b = -2, {V_1^3}_b = 8$ and $\mathring{E}_b = 4$ (see (8)). So $P_b = 5$ (around four internal edges there cannot be more or less than five hexes in an elementary hex mesh), so $P_a + P_b = 8$.
  - If ${V_1^3}_a = 2$,
    Then $P_a - P_b = -1$, ${V_1^3}_b = 6, \mathring{E}_a = 1$ and $\mathring{E}_b = 3$ (see (8)) so $P_a = 3$. With one internal edge, we have at least 3 internal faces and 3 hexes. Each hex has four $V_0^4$ (two with each of the other hexes) and the two $V_1^3$, then we can't have four $V_0^3$ on the same hex. This means that there is no way to add another hex (you can extend an elementary hex mesh by adding a hex to its boundary if and only if there is at least four $V_0^3$ on the same boundary's face). As $P_a - P_b = -1$ we have $P_b = 4$ which is absurd because $\mathring{E}_b = 3$ with $\mathring{V}_b = 0$.
  - If ${V_1^3}_a = 4$,
    then $P_a - P_b = 0, \mathring{E}_a = 2$. There is no way to have $P_a \neq 4$ because of two internal edges. So once more $P_a + P_b = 8$.
  - The cases ${V_1^3}_a = 6$ or 8 correspond to ${V_1^3}_b = 2$ or 0. They were already examined for ${V_1^3}_a$.

Finally, in an elementary hex mesh transformation into another elementary hex mesh with the boundary unchanged,

$$P_a + P_b = 8.$$

## 4 Flipping Quad and Hex Meshes

For Hexahedral and quadrilateral meshes, Bern and Eppstein's operations described in [1] as *flipping* for Hexahedral meshes and which are the extension in 3D of some of the quad refinement operations of Canann, Muthukrishnan and Phillips [4] (figure 2), can be use to modify locally hexes inside a mesh (figure 1). One property suggested in [1] for 2D flipping is that together with the parity-changing operation, it should form a complete set. Here we show that those flipping operations for quad or hex meshes are not linked.

**Theorem 5.** *[9]*
*Each set of flipping transformations [1] represented in figures 2 and 1 is topologically free; i.e., you can't go from one configuration to the other in a transformation using a combination of other transformations in a topological or geometrical mesh.*

*Proof.* The dual theory of mesh transformations and its sheet diagrams projections (see figure 3), [12], [13],[1] are more useful in this context, but it is not easy to use them in this case for relationships between different flipping operations. Our way is more algebraic.

Considering the number of new internal edges without a boundary vertex $n_e$, internal vertices $n_v$, internal faces without any boundary edge $n_f$, hexes $n_h$ in 3D and quads $n_q$ in 2D created or removed within each transformation, we model them as follows:

$$\begin{array}{lcll}
(0,0) \longleftrightarrow (0,0) & \Longrightarrow & \pm & (0_h, 0_f, 0_e, 0_v) \\
(1,1) \longleftrightarrow (1,1) & \Longrightarrow & \pm & (0_h, 0_f, 0_e, 0_v) \\
(1,0) \longleftrightarrow (0,1) & \Longrightarrow & \pm & (2_h, 0_f, 1_e, 2_v) \\
(2,0) \longleftrightarrow (0,2) & \Longrightarrow & \pm & (4_h, 1_f, 4_e, 4_v) \\
(3,0) \longleftrightarrow (0,3) & \Longrightarrow & \pm & (6_h, 6_f, 12_e, 8_v) \\
(2,1) \longleftrightarrow (1,2) & \Longrightarrow & \pm & (2_h, 0_f, 0_e, 0_v) \quad \text{in 3D,}
\end{array}$$

$$\begin{array}{lcll}
(0,0) \longleftrightarrow (0,0) & \Longrightarrow & \pm & (0_q, 0_e, 0_v) \\
(1,1) \longleftrightarrow (1,1) & \Longrightarrow & \pm & (0_q, 0_e, 0_v) \\
(1,0) \longleftrightarrow (0,1) & \Longrightarrow & \pm & (2_q, 1_e, 2_v) \\
(2,0) \longleftrightarrow (0,2) & \Longrightarrow & \pm & (4_q, 4_e, 4_v) \quad \text{in 2D.}
\end{array}$$

So if each set of flipping operations is not free, we can write one of the right right-hand side of the previous relationships as a linear combination of some others with coefficients in $\mathbb{Z}$. Let us look at these operations except $(0,0) \longleftrightarrow (0,0)$ and $(1,1) \longleftrightarrow (1,1)$ because those change nothing numerically; we have to solve the following equations in $\mathbb{Z}$ for each of the unknowns fixed at 1 or $-1$ (for example $x = 1$ means that you are going from the $(1,0)$ to the $(0,1)$ configuration and $x = -1$ is for the $(0,1)$ to the $(1,0)$ configuration). We obtain

$$\begin{cases} x + 2y + 3z + t = 0 \\ y + 6z = 0 \\ x + 4y + 12z = 0 \\ x + 2y + 4z = 0 \end{cases} \tag{10}$$

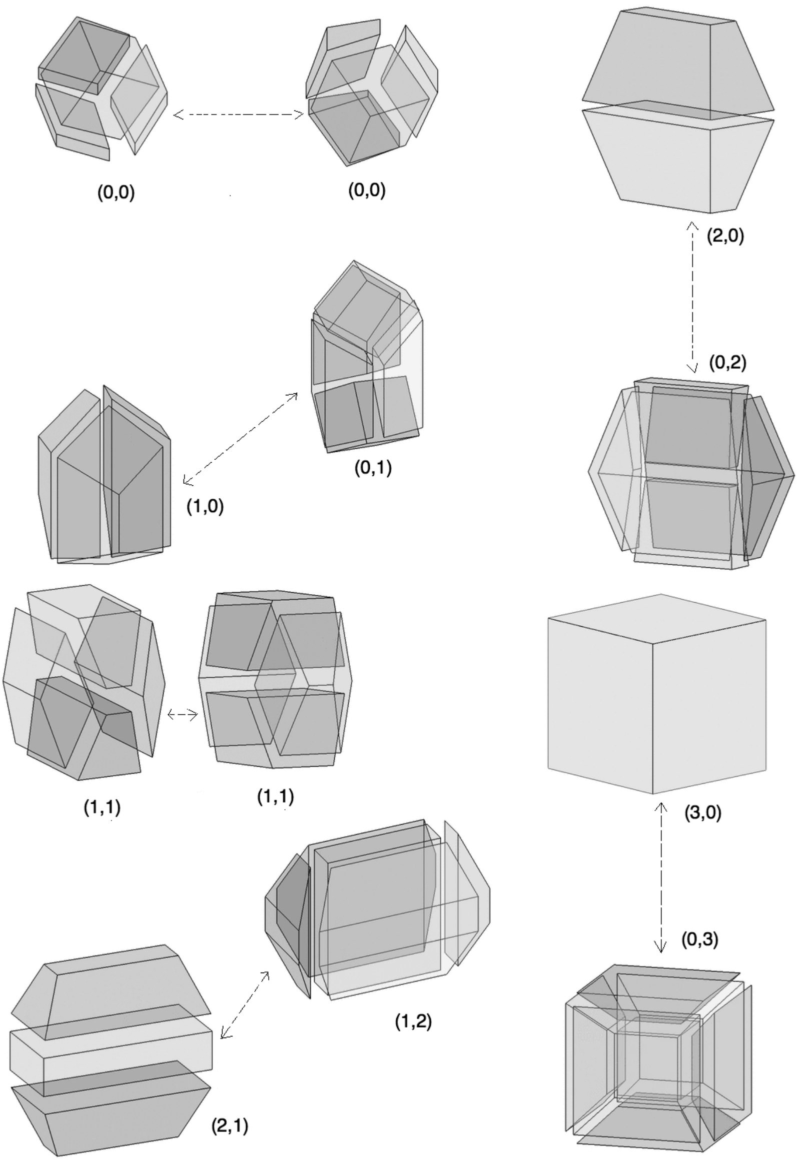


**Fig. 1.** The Flips for hex meshes

for the hexes flipping, and

$$\begin{cases} x + 2y = 0 \\ x + 4y = 0 \end{cases} \tag{11}$$

for the quads flipping.

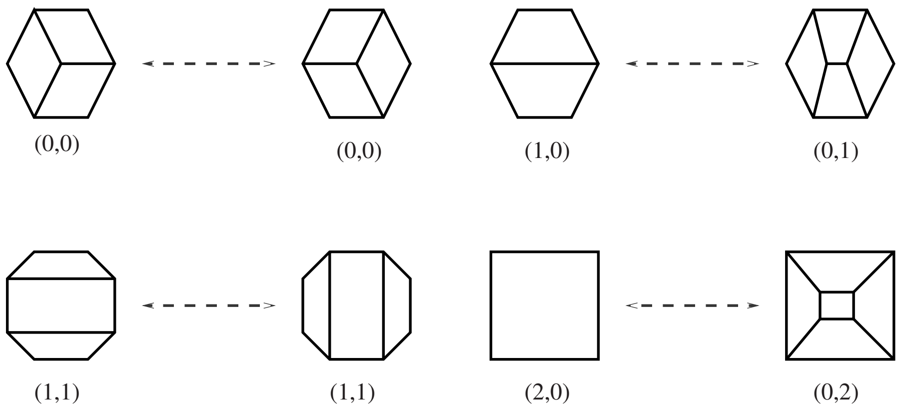


**Fig. 2.** The Flips for quad meshes

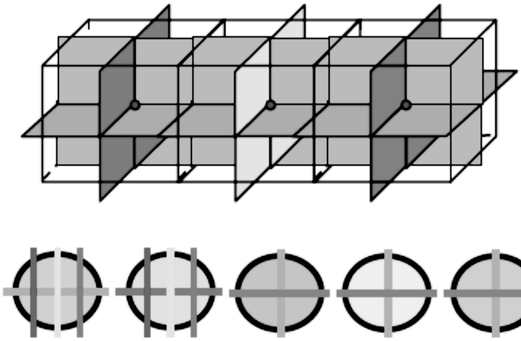

**Fig. 3.** Hex mesh with three elements (top); dual surfaces (sheets) and dual vertices shown. Two dimensional projection into "sheet diagrams" shown below.

For these two systems, $x = y = z = t = 0$ is the only solution. So there is no linear combination with those transformations. Then each set of operations without $(0,0) \longleftrightarrow (0,0)$ and $(1,1) \longleftrightarrow (1,1)$ is free. If we add the two other operation, it will not change the equations, nor their solutions. The possible solutions of those systems could have been considered as result of one of the two previous operations when combining all the others, so we can conclude that each set of flipping transformations is topologically free.

**Theorem 6.** *(existence of elementary mesh transformations)*
*The set of hexahedral flipping operations [1] figure 1 form the set of transformations that keep unchanged the boundary of any mesh in the subset of elementary hex meshes.*

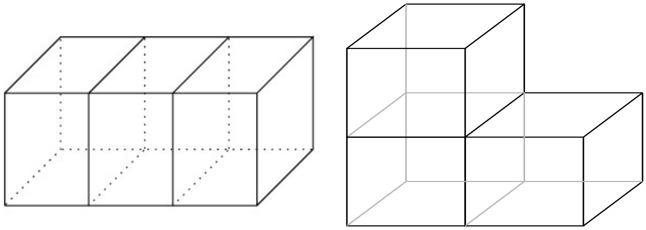

**Fig. 4.** some 3 hexes configurations

*Proof.* As the Theorem 4 says that any elementary mesh transformation needs at most 7 hexes and the sum of the hexes of the two configuration of the transformation is always 8, this is the case with all the operations of the set of hex flipping. Let us examine what happens for each number $P$ of hexes between 1 and 7:

- If $P = 1$ we have only one possibility, $\bar{F} = 6$ and it gives the $(3, 0) \longleftrightarrow (0, 3)$ flipping transformation.
- If $P = 2$ we have only one possibility, $\bar{F} = 10$ and it gives the $(2, 0) \longleftrightarrow (0, 2)$ flipping transformation.
- If $P = 3$ we have two possibilities:
  1. $\bar{F} = 12$ and it gives the $(1, 0) \longleftrightarrow (0, 1)$ flipping transformation.
  2. $\bar{F} = 14$ for the left configuration of the previous figure, there is no possible transformation without having five edge on a boundary vertex. But numerically, the solution of system (6) gives

$$P_b = 5, \quad \mathring{F}_b = 8, \ \mathring{V}_b = 0, \ \mathring{E}_b = 4$$

     which corresponds to the $(2, 1) \longleftrightarrow (1, 2)$ flipping transformation.
- If $P = 4$, we have $\bar{F} \leq 18$:
  1. If $\bar{F} = 18$, the four hexes should be aligned, the solution of system (6) tell us that the second configuration has a negative number of internal edges, so there is no possible transformation.
  2. If $\bar{F} = 16$, we will have $\mathring{V} = 1 - \frac{V_1^3}{2}$. It means that $\mathring{V} = 0$ or 2 any way $\mathring{V}$ will be negative in one of the two configurations, so there is no solution.
  3. If $\bar{F} = 14$, we obtain the $(1, 1) \longleftrightarrow (1, 1)$.
  4. If $\bar{F} = 12$, we obtain the $(0, 0) \longleftrightarrow (0, 0)$.
  5. If $\bar{F} \leq 10$, we will have $\mathring{F} \geq 7$. But a mesh with 4 hexes could not have more than 6 internal faces, so there is no solution.
- If $P = 5$ we have $\bar{F} \leq 22$ :
  1. if $22 \leq \bar{F} \leq 16$, there is one configuration where $\mathring{V}$ is negative.
  2. For the case 14 and 12, it corresponds to the solutions obtained with $P = 3$.
  3. As a mesh with 5 hexes couldn't have less than 10 boundary faces, if $\bar{F} = 10$ the second configuration will have 3 hexes with 10 boundary faces which is not realizable.

- If $P = 6$, we have $\bar{F} \leq 26$ :
  1. If $26 \leq \bar{F} \leq 12$ there is one configuration where $\mathring{V}$ is negative.
  2. For the case $\bar{F} = 10$, it corresponds to the solution obtained with $P = 2$.
  3. If $\bar{F} < 10$ the second configuration will have 2 hexes with less than 10 boundary faces which is not realizable.
- If $P = 7$, we have $\bar{F} \leq 30$ :
  1. If $30 \leq \bar{F} \leq 8$, there is one configuration where $\mathring{V}$ is negative.
  2. For the case 6, it corresponds to the solution obtained with $P = 1$.

So apart from the set of hex flips, there is no other transformation of an elementary hex mesh into another elementary hex mesh that keeps the boundary unchanged.

## 5 Some New Hex Mesh Local Modifications

We have previously observed that there is no other local hex transformation with an elementary hex mesh. So we are trying here to see what happens on a general hex mesh. Let us call $M_{e/v}$-hex mesh a hex mesh where the maximal number of edges per vertex is $M$.

We tried to build local hex transformations for a $5_{e/v}$-hex mesh and obtained four basic transformations: See figure 5, where $(x, y)$ denotes $x$ the number of boundary faces and $y$ the number of hexes. Those mesh files in medit [6] format, are available at `http://www.ann.jussieu.fr/~kuate/meshes.html`

These basic transformations can be extended by adding $k$ layers of suitable hexes, $k \in \mathbb{N}$, between the group of hexes on each side of those transformations (figure 6).

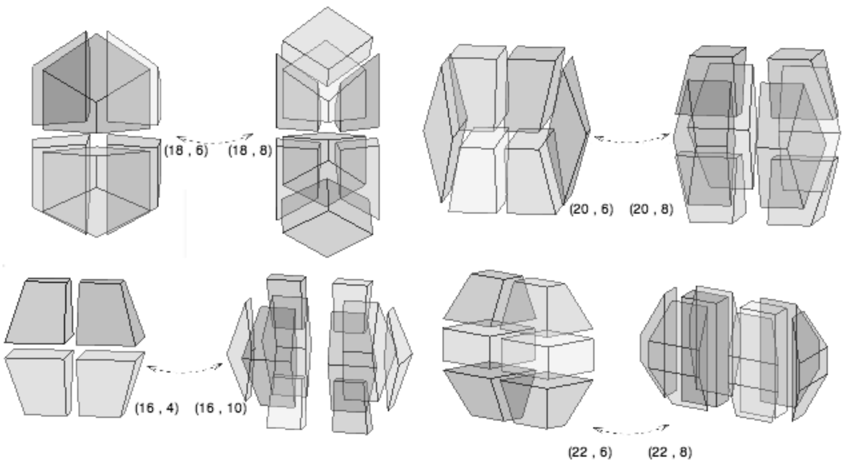


**Fig. 5.** Some basic $5_{e/v}$-hex meshes local transformations

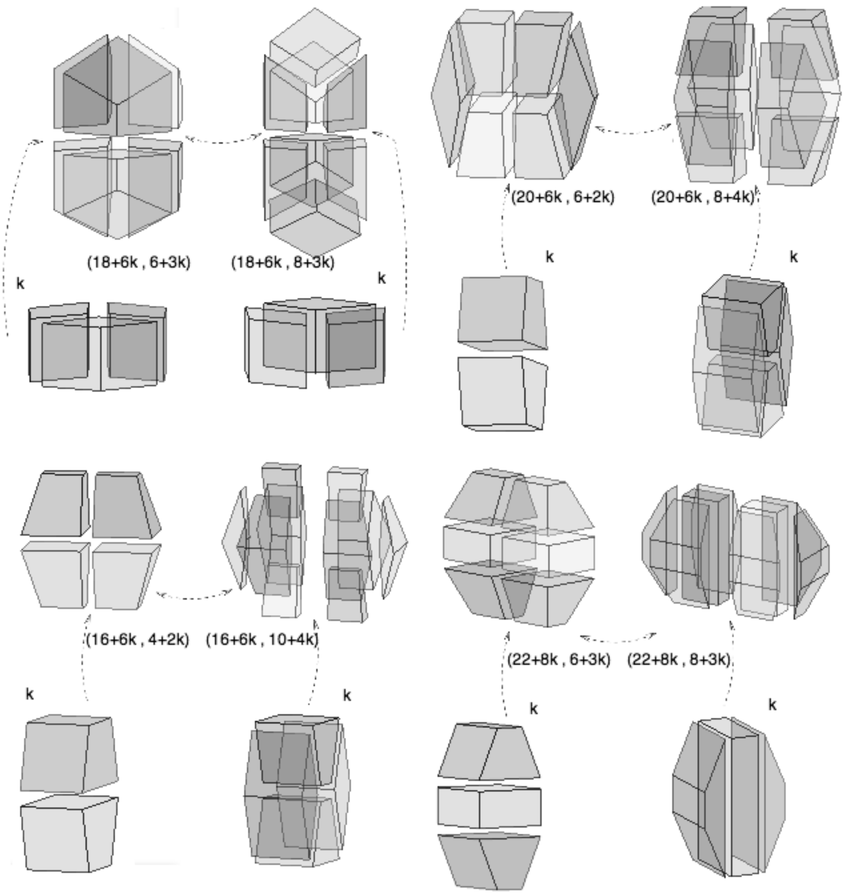


**Fig. 6.** Some $5_{e/v}$-hex meshes local transformations, $k \in \mathbb{N}$

**Theorem 7.** *For each fixed value of k, $k \in \mathbb{N}$, the set of four transformations obtained by adding k layers of suitable hexes on each configuration of the four basic $5_{e/v}$-hex meshes local transformations (figure 6), is topologically free.*

*Proof.* The same reasoning done in Theorem 5 gives the result.

## 6 Conclusion

We have described some properties of local transformations on quad meshes and essentially of the hex flipping transformations of Bern et. al. [1], from a geometrical and combinatorial point of view. This view allow us to propose a new set of basic local transformations for hexahedral meshes. We find that the sum of the number of hexes on both sides of each existing hexahedral local transformation equals $2 + 6k$, $k \in \mathbb{N}$, $k = 1$ corresponds to the set of Bern et. al. and $k \geq 2$ to ours. Then all those geometrical transformations cannot change the parity of the mesh. It would be interesting to prove that the previous relationship is the only one that is possible to have on any geometrical hex mesh with less than 6 edges per vertex. Therefore the problem of parity changing on a geometrical hex mesh in more complicated situations (some vertices should have more than 5 edges) may be considered.

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