Research Note
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Smoothing of Unstructured Grids

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Abstract

Two mesh smoothing techniques applied to unstructured grids are compared. These are based on the solution of elliptic equations, Laplace and Winslow, applied to a uniform grid in computational space. In the first case, the equations are solved by a barycentric averaging procedure, and for the second case, a vertex-based finite volume scheme, with piece-wise local virtual control volumes is used. In addition, an improved treatment for cross derivatives terms in the Winslow equations has been implemented.

The smoothing characteristics of these two techniques are compared using three quality measures: minimum angle, shape factor and a smoothness ratio as local criteria. Finally, a global quality smoothness criterion is introduced and used to assess global smoothing properties of these methods.

Keywords: Elliptic smoothing, Winslow Equation, Barycentric method, Local computational space, Unstructured grids, Mesh generation;

1. Introduction

The solution of Winslow's equation is a widely used procedure for the generation or the smoothing of grids. The approach consists of mapping an isotropic grid in computational space onto an arbitrary domain in physical space. This is carried out as the solution of a boundary value problem where the target shape in the physical domain, Ω, is imposed by the body coordinates through the boundary conditions of the PDE solved in computational space, C.

Initially proposed by Winslow [1], this has been extensively applied for the generation of structured grids [2], where the computational mesh is an implied cartesian mesh in (ξ, η) space.

$$\begin{cases} \mathcal{L}(x) = g_{11}x_{\xi\xi} - 2g_{12}x_{\xi\eta} + g_{22}x_{\eta\eta} = 0, \\ \mathcal{L}(y) = g_{11}y_{\xi\xi} - 2g_{12}y_{\xi\eta} + g_{22}y_{\eta\eta} = 0, \end{cases} \quad (1)$$

where,

$$\begin{aligned} g_{11} &= x_\eta^2 + y_\eta^2, \\ g_{12} &= x_\xi x_\eta + y_\xi y_\eta, \\ g_{22} &= x_\xi^2 + y_\xi^2, \end{aligned}$$

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Hence, a grid in physical space is created through a transformation of a computational mesh by the numerical solution of Eq. 1 or the use of functionals [3], discretized by second order finite differences or finite volume schemes, for structured or unstructured grids, respectively. The extension of this procedure to unstuctured grids encounters two difficulties due to non-conservation form of these equations, as well as the lack of an implied unique computational domain. Knupp [4] has shown unstructured Winslow mesh smoothing on unstructured quadrilateral meshes using a locally defined computational domain. Karman[5] and Sahasrabudhe[6] successfully applied unstructured smoothing by introducing local optimized computational domains called "virtual control volumes". In the current work, this method, with an improved averaging procedure in order to handle cross derivative terms, has been implemented.

2. Grid generation as a mapping

Arabi et al.[7] addressed two major concerns regarding the application of elliptic smoothing to unstructured meshes. The first is the non-conservative formulation of the basic equations, and the second is the mapping procedure by employing an explicit mapping. The domain shape is mapped to a computational space where a uniform unstructured grid is generated. This mesh is then mapped to physical space by the numerical solution of Eq. 1 using a finite volume method. Using a linearization procedure, Eqs. 1 can be integrated over a control volume defined around each vertex of the mesh in computational space. The integration path for the application of Green's theorem is formed by joining the centroid of each triangular element to the midpoints of its sides, as shown by the dashed lines in Fig. 1. These divide each triangular element into three equal areas, which collectively, form non-overlapping contiguous control volumes associated with a vertex in the mesh.

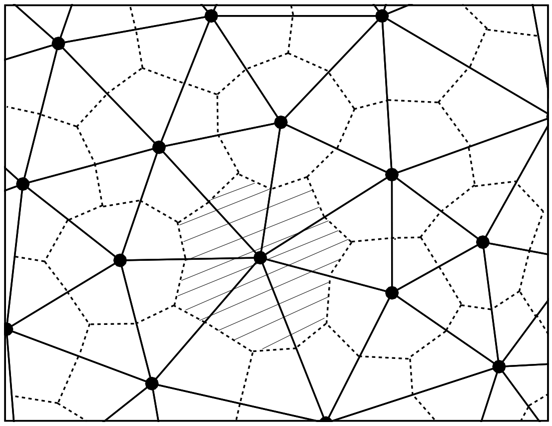
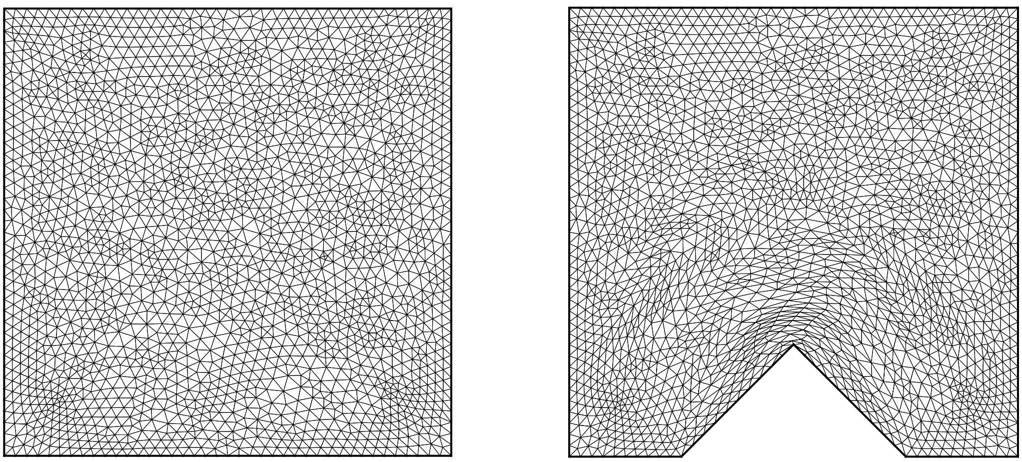


Fig. 1. Control volume for each vertex (dashed lines) of the unstructured triangular mesh (solid lines)

An example of the application of this approach, Fig. 2 illustrates the mapping of a uniform unstructured mesh in computational space, C to a sharp wedge in Ω.



(a) Mesh in computational space C

(b) Mesh mapped to physical space Ω

Fig. 2. Mapping of an unstructured mesh in computational domain to a spike in physical space using Winslow's equation.

3. Elliptic Smoothing

Grid smoothing is a post-processing procedure designed to improve the mesh quality of an existing grid, and as such this operation follows the mesh generation step. This can be done with various techniques where the nodal coordinates of an unstructured mesh are modified as the solution of an operator such as Laplace, Winslow[1], or the use of functionals [3]. It consists in solving a partial differential equation, where the dependent variables are the coordinates in physical space, (x, y), in terms of independent variables in computational space (ξ, η). This is appropriate for unstructured grids, but unlike structured grids the numerical techniques described in Sect. 2 are not directly applicable. Specifically, the computational space is no longer an implied cartesian grid, and needs to be specified explicitly.

Elliptic equations enforce a smoothness condition by using a uniform spacing in the computational domain (ξ, η). In structured meshes, the computational domain has the same topology as the physical space, and ideal spacing i.e $\delta\xi = \delta\eta$, thus effectively enforcing an equal spacing around each node.

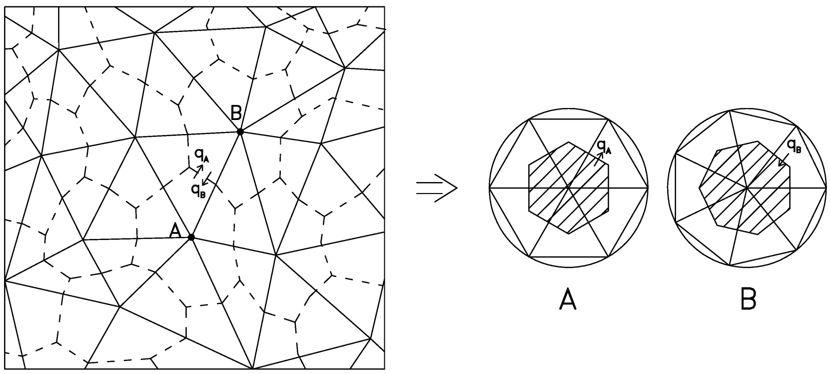


Fig. 3. Local mapping from physical to computational space

For an unstructured mesh, there is no such a universal computational domain that matches every unstructured topology. As argued by [5], one obvious way of obtaining such a computational mesh would be to simply copy the existing physical mesh. However, if both the physical and computational meshes are identical, the smoothness condition is satisfied, almost by definition, and no nodes are moved. A solution to this problem was proposed by Sahasrabudhe [8] with the introduction of computational domains that are only defined locally. For a stencil of elements surrounding a single node, an ideal mesh can be defined by equally spacing the connected points on a unit circle. These stencils, called virtual control volumes, are locally defined and can be used to drive the solution of the smoothing equations. Thus formulated, smoothing by the use of elliptic operators becomes a set of distinct boundary value problems with dirichlet boundary conditions for each node.

4. Mesh Quality and Mesh Smoothness

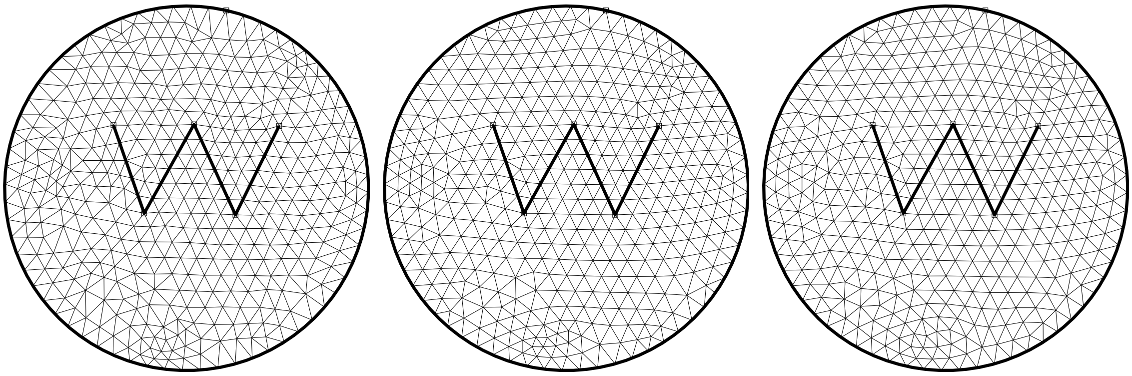
Three criteria were used to measure the quality of a mesh: minimum angle, shape factor and smoothness,

Minimum angle	Shape factor	Smoothness ratio
$min(\alpha_n) \quad n = 1:3$	$SQ_i = \frac{4\sqrt{3}A_i}{\sum_{i=1}^{3} l_i^2}$	$SF_i = \frac{A_i}{max(A_n)}$

The shape factor criterion measures the likeness of an element to a reference equilateral triangle where A_i is the area of the triangle, and $l_i (i = 1, 2, 3)$ are the lengths of the triangle's edges. The smoothness criterion was introduced by [9] where SR_i represents the smoothness ratio, and the denominator represents the maximum area of its adjacent cells. An ideal values for SR_i is as close as possible to one.

5. Results and Discussion

Elliptic smoothing based on Winslow's equation was implemented based on the finite volume techniques using piecewise local mapping to a unit circle in computational space. This was validated through numerous test cases for different geometries and grid sizes, and compared to the more classical Laplace smoothing. Results for a representative configuration, a "W" slit inside a circle, will be used to illustrate the results. The effect of both smoothing techniques on an initial raw mesh, generated using a frontal unstructured grid procedure, is shown in Fig.4.



(a) Raw Mesh (b) Barycentric (c) Winslow

Fig. 4. Comparison of the raw mesh and two smoothed grids

These qualitative results are quantified in Fig.5 which give the distribution of the minimal angle, the shape measure, and smoothness ratio, for the barycentric averaging procedure and the Winslow smoothing, respectively. It can be observed that for the first two criteria, the barycentric method gives a more satisfactory distribution. However, from smoothness point of view, it is clear that the Winslow operator gives better results than the barycentric method.

In contrast to the traditional definition of mesh quality, which considers individual criteria of each element, smoothness can be defined globally as the Smoothness Factor (SF) of the entire mesh, as follows,

$$SF = \frac{1}{N_e}\sum_{1}^{N_e} min(SR_i, \frac{1}{SR_i}) \qquad (2)$$

where N_e is the total number of elements in the mesh. The range of values for this factor is $0 < SF \leq 1$, and hence, the greater SF, the smoother the mesh.

The following table shows that the global smoothness for Winslow's method is higher than that resulting from the barycentric method.

Operator	Raw Mesh	Barycentric	Winslow
Global smoothness	0.865	0.913	0.948

6. Conclusion

Two elliptic mesh smoothing methods, barycentric and Winslow, have been compared for 2D unstructured grids. While both methods show clear improvement for all three mesh quality criteria over the raw grid, the barycentric method almost always gives equal or better results than the Winslow equations. This unexpected behaviour can be attributed to the use of disjoint mappings of control volumes around each vertex to a unit circle which may not strickly constitute a formal solution of Winslow's operator as the control volumes overlap and as there is no continuity of the fluxes between adjoining control volumes. This makes the method a suitable averaging process while it is not still a finite volume solution of Winslow equation.

However, the Winslow operator always shows better results for the global mesh smoothness.

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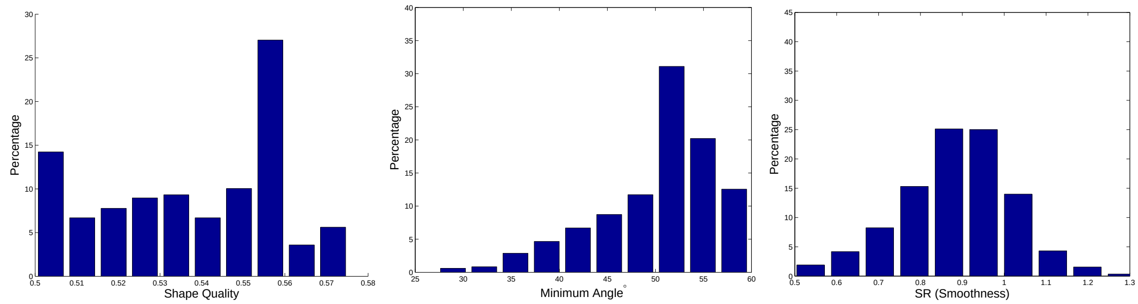
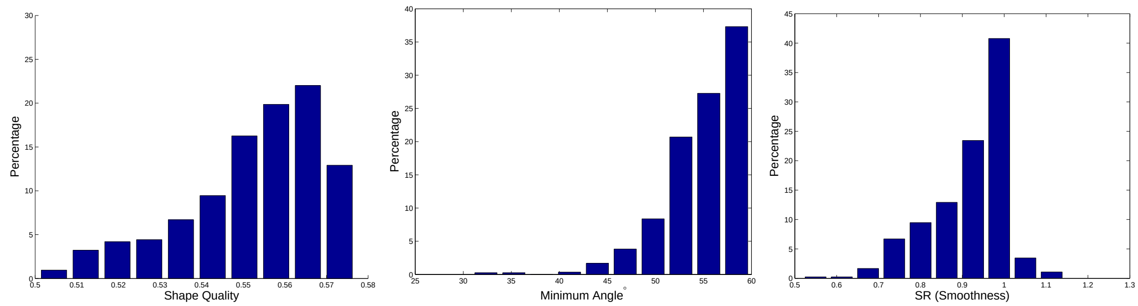
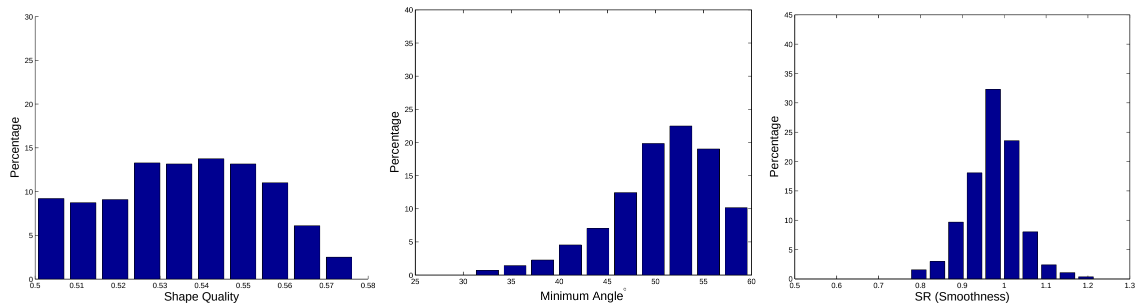


(d) Raw Mesh



(i) Barycentric



(o) Winslow

Fig. 5. Comparison of mesh quality and smoothness for a discontinuous slit inside a circle for the Raw mesh, Barycentric and Winslow smoothing

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